Supplementary Paper for:

Nonparametric Testing for Differences in Electricity Prices: The Case of the Fukushima Nuclear Accident

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Our basic assumptions, are essentially equivalent to those in Ruppert and Wand (1994) with some straightforward adjustments to our functional data and time series context.

A1 (Asymptotic Scenario) $nm \to \infty$, where $m = m_n \geq 2$ such that $m_n \asymp n^\theta$ with $0 \leq \theta < \infty$. Hereby, “$m_n \asymp n^\theta$” denotes that the two sequences $m_n$ and $n^\theta$ are asymptotically equivalent, i.e., that $\lim_{n \to \infty} (m_n / n^\theta) = C$ with constant $0 < C < \infty$.

A2 (Random Design) The triple $(Y_{ij}, U_{ij}, Z_i)$ has the same distribution as $(Y, U, Z)$ with pdf $f_{YUZ}$ where $f_{YUZ}(y, u, z) > 0$ for all $(y, u, z) \in \mathbb{R} \times [0, 1]^2$ and zero else. The error term $\epsilon_{ij}$ is iid and independent from $X^c_i$, $U^s$, and $Z_s$ for all $s = 1, \ldots, n$ and $\ell = 1, \ldots, m$.

A3 (Smoothness & Kernel) The pdf $f_{YUZ}(y, u, z)$ and its marginals are continuously differentiable. All second-order derivatives of the function $\mu$ are continuous. The (auto-)covariance functions $\gamma_l((u_1, z_1), (u_2, z_2)) = \mathbb{E}(X^c_i(u_1, z_1)X^c_{i+l}(u_2, z_2))$, $l \geq 0$, are continuously differentiable for all points within their supports. The multiplicative kernel functions $K_\mu$ and $K_\gamma$ are products of second-order kernel functions $\kappa$.

A4 (Moments & Dependency) $X_i$, $U_{ij}$, and $Z_i$ are strictly stationary, ergodic, and weakly dependent time series with auto-covariances that converge uniformly to zero at a geometrical rate. It is assumed that $\mathbb{E}(X_i(u, z)^4) < \infty$, $\mathbb{E}(\epsilon_{ij}) = 0$, $\mathbb{E}(\epsilon_{ij}^2) = \sigma^2_\epsilon < \infty$ for all $(u, z)$, $i$, and $j$.

A5 (Bandwidths) $h_{\mu, U}, h_{\mu, Z} \to 0$ and $(nm)h_{\mu, U}h_{\mu, Z} \to \infty$ as $nm \to \infty$. $h_{\mu, U}, h_{\mu, Z} \to 0$ and $(nM)h_{\mu, U}^2h_{\mu, Z} \to \infty$ as $nM \to \infty$.

Remark. Assumption A1 is a simplified version of the asymptotic setup of Zhang and Wang (2016). The case $\theta = 0$ implies that $m$ is bounded which corresponds to a simplified version of the finite-$m$ asymptotic considered by Jiang and Wang (2010). For $0 < \theta < \infty$ we can consider all further scenarios from sparse to dense functional data. In line with our real data application,
we consider a deterministic $m$ as also done, for instance, by Hall, Müller and Wang (2006). However, our results are generalizable to a random $m$ using some minor modifications.

APPENDIX A: PROOFS

The following Lemma A.1 builds the basis of our theoretical results.

**Lemma A.1 (Bias and Variance of $\hat{\mu}$).** Let $(u, z)$ be an interior point of $[0,1]^2$. Under Assumptions A1-A5 the conditional asymptotic bias and variance of the LLK estimator $\hat{\mu}$ in Eq. (2.2) are then given by

(i) Bias $\{\hat{\mu}(u, z; h_{\mu,U}, h_{\mu,Z})|U, Z\} = B_\mu(u, z) + o_p(h_{\mu,U}^2 + h_{\mu,Z}^2)$ with

$B_\mu(u, z) = \frac{1}{2} \nu_2(K_\mu) \left( h_{\mu,U}^2 \mu^{(2,0)}(u, z) + h_{\mu,Z}^2 \mu^{(0,2)}(u, z) \right)$, where

$\mu^{(k,l)}(u,z) = \left( \frac{\partial}{\partial u}^k \frac{\partial}{\partial z}^l \right) \mu(u,z)$.

(ii) Variance $\{\hat{\mu}(u, z; h_{\mu,U}, h_{\mu,Z})|U, Z\} = (V_\mu^I(u, z) + V_\mu^II(u, z)) \left( 1 + o_p(1) \right)$ with

$V_\mu^I(u, z) = \frac{1}{nm} \left[ h_{\mu,U}^{-1} h_{\mu,Z}^{-1} R(K_\mu) \gamma(u, u, z) + \sigma_e^2 f_{UZ}(u, z) \right]$ and

$V_\mu^II(u, z) = \frac{n}{m} \left[ h_{\mu,Z}^{-1} R(\kappa) \frac{\gamma(u, u, z)}{f_Z(z)} \right]$.

**Proof of Lemma A.1.** Our proof of Lemma A.1 generally follows that of Ruppert and Wand (1994), and differs only from the latter reference as we consider additionally a conditioning variable $Z_i$, a function-valued error term, and a time series context.

Proof of Lemma A.1, part (i): For simplicity, consider a second-order kernel function $\kappa$ with compact support such as the Epanechnikov kernel; this is, of course, without loss of generality. Let $(u, z)$ be a interior point of $[0,1]^2$ and define $H_\mu = \text{diag}(h_{\mu,U}^2, h_{\mu,Z}^2)$, $U = (U_{11}, \ldots, U_{nm})^\top$, and $Z = (Z_1, \ldots, Z_n)^\top$. Using a Taylor-expansion of $\mu$ around $(u, z)$, the conditional bias of the estimator $\hat{\mu}(u, z; H)$ can be written as

$$E(\hat{\mu}(u, z; H_\mu) - \mu(u, z)|U, Z) =$$

$$= \frac{1}{2} e_1^\top \left( (nm)^{-1} [1, U_u, Z_z]^\top W_{\mu,uz} |1, U_u, Z_z] \right)^{-1} \times$$

$$\times (nm)^{-1} [1, U_u, Z_z]^\top W_{\mu,uz} (Q_\mu(u, z) + R_\mu(u, z)),$$

where $Q_\mu(u, z)$ is a $nm \times 1$ vector with typical elements

$$(U_{ij} - u, Z_i - z) H_\mu(u, z)(U_{ij} - u, Z_i - z)^\top \in \mathbb{R}$$
with $\mathcal{H}_\mu(u, z)$ being the Hessian matrix of the regression function $\mu(u, z)$. The $nm \times 1$ vector $R_\mu(u, z)$ holds the remainder terms as in Ruppert and Wand (1994).

Next we derive asymptotic approximations for the $3 \times 3$ matrix

\[
((nm)^{-1}[1, U_u, Z_z]^T W_{\mu, uz}[1, U_u, Z_z])^{-1}
\]

and the $3 \times 1$ matrix

\[
((nm)^{-1}[1, U_u, Z_z]^T W_{\mu, uz} Q_\mu(u, z)
\]

of the right hand side of Eq. (A.1). Using standard arguments from nonparametric statistics it is easy to derive that

\[
((nm)^{-1}[1, U_u, Z_z]^T W_{\mu, uz}[1, U_u, Z_z])^{-1} =
\]

\[
\begin{pmatrix}
\nu_2(K_\mu) H_{\mu} f_{UZ}(u, z) + o_p(1) & \nu_2(K_\mu) f_{UZ}(u, z) H_{\mu} + o_p(1^T H_{\mu}) \\
\nu_2(K_\mu) H_{\mu} f_{UZ}(u, z) + o_p(H_{\mu} 1) & \nu_2(K_\mu) f_{UZ}(u, z) + o_p(H_{\mu})
\end{pmatrix},
\]

where $1 = (1, 1)^T$ and $D_{f_{UZ}}(u, z)$ is the vector of first order partial derivatives (i.e., the gradient) of the pdf $f_{UZ}$ at $(u, z)$. Inversion of the above block matrix yields

\[
((nm)^{-1}[1, U_u, Z_z]^T W_{\mu, uz}[1, U_u, Z_z])^{-1} =
\]

\[
\begin{pmatrix}
(f_{UZ}(u, z))^{-1} + o_p(1) & -D_{f_{UZ}}(u, z)(f_{UZ}(u, z))^{-2} + o_p(1) \\
-D_{f_{UZ}}(u, z)(f_{UZ}(u, z))^{-2} + o_p(1) & (\nu_2(K_\mu) H_{\mu} f_{UZ}(u, z))^{-1} + o_p(H_{\mu})
\end{pmatrix}.
\]

The $3 \times 1$ matrix $((nm)^{-1}[1, U_u, Z_z]^T W_{\mu, uz} Q_\mu(u, z)$ can be partitioned as follows:

\[
((nm)^{-1}[1, U_u, Z_z]^T W_{\mu, uz} Q_\mu(u, z) = \begin{pmatrix}
\text{upper element} \\
\text{lower bloc}
\end{pmatrix},
\]

where the $1 \times 1$ dimensional upper element can be approximated by

\[
((nm)^{-1}\sum_{ij} K_{\mu, h}(U_{ij} - u, Z_i - z)(U_{ij} - u, Z_i - z)\mathcal{H}_\mu(u, z)(U_{ij} - u, Z_i - z)^T
\]

\[
= (\nu_2(\kappa))^2 \text{tr} \{H_{\mu} \mathcal{H}_\mu(u, z)\} f_{UZ}(u, z) + o_p(\text{tr}(H_{\mu}))
\]

and the $2 \times 1$ dimensional lower bloc is equal to

\[
((nm)^{-1}\sum_{ij} \{K_{\mu, h}(U_{ij} - u, Z_i - z)(U_{ij} - u, Z_i - z)\mathcal{H}_\mu(u, z)(U_{ij} - u, Z_i - z)^T\} \times
\]

\[
(U_{ij} - u, Z_i - z)^T = O_p(H_{\mu}^{3/2} 1).
\]
Plugging the approximations of Eqs. (A.2)-(A.4) into the first summand of the conditional bias expression in Eq. (A.1) leads to the following expression

$$
\frac{1}{2} e_1^\top ((nm)^{-1}[1, U_u, Z_u] W_{\mu, uz}[1, U_u, Z_u])^{-1} \times \\
\times (nm)^{-1}[1, U_u, Z_u] W_{\mu, uz} Q_\mu(u, z) = \\
= \frac{1}{2} (\nu_2(\kappa))^2 tr \{H_\mu H_\mu(u, z)\} + o_p(tr(H_\mu)).
$$

Furthermore, it is easily seen that the second summand of the conditional bias expression in Eq. (A.1), which holds the remainder term, is given by

$$
\frac{1}{2} e_1^\top ((nm)^{-1}[1, U_u, Z_u] W_{\mu, uz}[1, U_u, Z_u])^{-1} \times \\
\times (nm)^{-1}[1, U_u, Z_u] W_{\mu, uz} R_\mu(u, z) = o_p(tr(H_\mu)).
$$

Summation of the two latter expressions yields the asymptotic approximation of the conditional bias

$$
E(\hat{\mu}(u, z; H_\mu) - \mu(u, z)|U, Z) = \frac{1}{2} (\nu_2(\kappa))^2 tr \{H_\mu H_\mu(u, z)\} + o_p(tr(H_\mu)).
$$

Proof of Lemma A.1, part (ii): In the following we derive the conditional variance of the local linear estimator $V(\hat{\mu}(u, z; H_\mu)|U, Z) =

= e_1^\top [(1, U_u, Z_u) W_{\mu, uz}[1, U_u, Z_u])^{-1} \times \\
\times [1, U_u, Z_u] W_{\mu, uz} \text{Cov}(Y|U, Z) W_{\mu, uz}[1, U_u, Z_u] \times \\
\times [(1, U_u, Z_u) W_{\mu, uz}[1, U_u, Z_u])^{-1} e_1 
$$

(A.5)

$$
= e_1^\top ((nm)^{-2}[1, U_u, Z_u] W_{\mu, uz}[1, U_u, Z_u])^{-1} \times \\
\times ((nm)^{-2}[1, U_u, Z_u] W_{\mu, uz} \text{Cov}(Y|U, Z) W_{\mu, uz}[1, U_u, Z_u]) \times \\
\times ((nm)^{-1}[1, U_u, Z_u] W_{\mu, uz}[1, U_u, Z_u])^{-1} e_1,
$$

where $\text{Cov}(Y|U, Z)$ is a $nm \times nm$ matrix with typical elements

$$
\text{Cov}(Y_{ij}, Y_{ik}|U_{ij}, U_{ik}, Z_i, Z_k) = \gamma_{i-j}(\{(U_{ij}, Z_i), (U_{ik}, Z_k)\}) + \\
+ \sigma^2(\ell, \ell) 1(i = \ell, j = \ell);
$$

with $1(.)$ being the indicator function.

We begin with analyzing the $3 \times 3$ matrix

$$
(nm)^{-2}[1, U_u, Z_u] W_{\mu, uz} \text{Cov}(Y|U, Z) W_{\mu, uz}[1, U_u, Z_u]
$$

using the following three Lemmas A.2-A.4.
LEMMA A.2. The upper-left scalar (block) of the matrix $(nm)^{-2}[1, U, Z] W_{\mu,u,z} \text{Cov}(Y|U, Z) W_{\mu,u,z}[1, U, Z]$ is given by

$$(nm)^{-2}1^T W_{\mu,u,z} \text{Cov}(Y|U, Z) W_{\mu,u,z} 1$$

$$= (nm)^{-1} f_{UZ}(u,z) |H_\mu|^{-1/2} R(K_\mu) (\gamma(u,u,z) + \sigma_1^2) (1 + O_p(tr(H_\mu^{1/2})))$$

$$+ n^{-1} (f_{UZ}(u,z))^2 (m - 1) h_{\mu,Z}^{-1} R(\kappa) \gamma(u,u,z) f_Z(z) + c(u,z) (1 + O_p(tr(H_\mu^{1/2})))$$

$$= O_p((nm)^{-1}|H_\mu|^{-1/2} + O_p(n^{-1}h_{\mu,Z}^{-1}),$$

where $c(u,z) = 2 \sum_{l=1}^{n-1} \gamma_l((u,z),(u,z))$. Under Assumption A4 there exists a constant $C$, $0 < C < \infty$, such that $0 \leq |c(u,z)| \leq C$.

LEMMA A.3. The $1 \times 2$ dimensional upper-right block of the matrix $(nm)^{-2}[1, U, Z] W_{\mu,u,z} \text{Cov}(Y|U, Z) W_{\mu,u,z}[1, U, Z]$ is given by

$$(nm)^{-2}1^T W_{\mu,u,z} \text{Cov}(Y|U, Z) W_{\mu,u,z}$$

$$\begin{pmatrix} (U_{11} - u, Z_1 - z) \\
\vdots \\
(U_{nm} - u, Z_n - z) \end{pmatrix}$$

$$= (nm)^{-1} f_{UZ}(u,z) |H_\mu|^{-1/2}(1^T H_\mu^{1/2}) R(K_\mu) (\gamma(u,u,z) + \sigma_1^2) (1 + O_p(tr(H_\mu^{1/2})))$$

$$+ n^{-1} (f_{UZ}(u,z))^2 (1^T H_\mu^{1/2}) (m - 1) h_{\mu,Z}^{-1} R(\kappa) \gamma(u,u,z) f_Z(z) + c(u,z) (1 + O_p(tr(H_\mu^{1/2})))$$

$$= O_p((nm)^{-1}|H_\mu|^{-1/2}(1^T H_\mu^{1/2}) + O_p(n^{-1}(1^T H_\mu^{1/2})h_{\mu,Z}^{-1},$$

where $c(u,z) = 2 \sum_{l=1}^{n-1} \gamma_l((u,z),(u,z))$. Under Assumption A4 there exists a constant $C$, $0 < C < \infty$, such that $0 \leq |c(u,z)| \leq C$.

Remark. The $2 \times 1$ dimensional lower-left block of the matrix $(nm)^{-2}[1, U, Z] W_{\mu,u,z} \text{Cov}(Y|U, Z) W_{\mu,u,z}[1, U, Z]$ is simply the transposed version of the result in Lemma A.3.
Lemma A.4. The $2 \times 2$ lower-right block of the matrix $(nm)^{-2} [1, U, Z] \trans W_{\mu,uz} \Cov(\Y|\U, \Z) W_{\mu,uz} [1, U, Z] \trans$ is given by

$$(nm)^{-2} ((U_{11} - u), (Z_{1} - z)) \trans, \ldots, ((U_{nm} - u), (Z_{n} - z)) \trans) \times$$

$$W_{\mu,uz} \Cov(\Y|\U, \Z) W_{\mu,uz} 
\begin{pmatrix}
(U_{11} - u, Z_{1} - z) \\
\vdots \\
(U_{nm} - u, Z_{n} - z)
\end{pmatrix}$$

$$= (nm)^{-1} f_{UZ}(u, z) |H_{\mu}|^{-1/2} H_{\mu} R(K_{\mu}) \left( \gamma(u, u, z) + \sigma_{\gamma}^2 \right) \left( 1 + O_{p}(\text{tr}(H_{\mu}^{1/2})) \right)$$

$$+ n^{-1} \left( f_{UZ}(u, z) \right)^{2} H_{\mu} \left[ \left( \frac{m - 1}{m} \right) h_{\mu, z}^{-1} R(\kappa) \frac{\gamma(u, u, z)}{f_{Z}(z)} + c_{r} \right] \left( 1 + O_{p}(\text{tr}(H_{\mu}^{1/2})) \right)$$

$$= O_{p}((nm)^{-1}|H_{\mu}|^{-1/2} H_{\mu}) + O_{p}(n^{-1} H_{\mu}^{-1/2} H_{\mu})$$,

where $c(u, z) = 2 \sum_{i=1}^{n-1} \gamma((u, z), (u, z))$. Under Assumption A4 there exists a constant $C$, $0 < C < \infty$, such that $0 \leq |c(u, z)| \leq C$.

Using the approximations for the bloc-elements of the matrix $(nm)^{-2} [1, U, Z] \trans W_{\mu,uz} \Cov(\Y|\U, \Z) W_{\mu,uz} [1, U, Z] \trans$, given by the Lemmas A.2-A.4, and the approximation for the matrix $((nm)^{-1} [1, U, Z] \trans W_{\mu,uz} [1, U, Z] \trans)^{-1}$, given in (A.2), we can approximate the conditional variance of the bivariate local linear estimator, given in (A.5). Some straightforward matrix algebra leads to $V(\hat{\mu}(u, z; H_{\mu})|\U, \Z) = (nm)^{-1}|H_{\mu}|^{-1/2}$

$$R(K_{\mu}) \left( \gamma(u, u, z) + \sigma_{\gamma}^2 \right) \left( 1 + o_{p}(1) \right)$$

$$+ n^{-1} \left[ \left( \frac{m - 1}{m} \right) h_{\mu, z}^{-1} R(\kappa) \frac{\gamma(u, u, z)}{f_{Z}(z)} + c_{r} \right] \left( 1 + o_{p}(1) \right)$$,

which is asymptotically equivalent to our variance statement of Lemma A.1 part (ii).

Proof of Lemma A.2. (The proofs of Lemmas A.3 and A.4 can be done correspondingly.) To show Lemma A.2 it will be convenient to split the sum such that $(nm)^{-2} 1 \trans W_{\mu,uz} \Cov(\Y|\U, \Z) W_{\mu,uz} 1 = s_{1} + s_{2} + s_{3}$. Using standard procedures from kernel density estimation leads to

$$s_{1} = (nm)^{-2} \sum_{ij} (K_{\mu,h}(U_{ij} - u, Z_{i} - z))^{2} V(Y_{ij}|\U, \Z)$$

$$= (nm)^{-1}|H_{\mu}|^{-1/2} f_{UZ}(u, z) R(K_{\mu}) \left( \gamma(u, u, z) + \sigma_{\gamma}^2 \right) + O((nm)^{-1}|H_{\mu}|^{-1/2} \text{tr}(H_{\mu}^{1/2}))$$

(A.6)

$$s_{2} = (nm)^{-2} \sum_{jk} \sum_{it} K_{\mu,h}(U_{ij} - u, Z_{i} - z) \Cov(Y_{ij}, Y_{ik}|\U, \Z) K_{\mu,h}(U_{ik} - x, Z_{t} - z)$$

$$= n^{-1} (f_{UZ}(u, z))^{2} c(u, z) + O_{p}(n^{-1} \text{tr}(H_{\mu}^{1/2}))$$

(A.7)
s_3 = \frac{(nm)^{-2}}{\sum_{ij}^{m} \sum_{t}^{T} h_{\mu,U}^{-1}(U_{ij} - u)(h_{\mu,Z}^{-1}(Z_t - z))^2 \text{Cov}\left(Y_{ij}, Y_{jt} | U, Z\right)} \times h_{\mu,U}^{-1}(U_{ik} - x) \\
= n^{-1}(f_{UZ}(u, z))^2 \left[ \left( \frac{m-1}{m} \right) h_{\mu,Z}^{-1} R(\kappa) \frac{\gamma(u, u, z)}{f_Z(z)} \right] + O_p(n^{-1} \text{tr}(H_{\mu}^{1/2})),

where \( c(u, z) = 2 \sum_{l=1}^{n} \gamma_l((u, z), (u, z)) \). Summing up (A.6)-(A.7) leads to the result in Lemma A.2. Lemmas A.3 and A.4 differ from Lemma A.2 only with respect to the additional factors \( \mathbf{1}^\top H_{\mu}^{1/2} \) and \( H_{\mu} \). These come in due to the usual substitution step for the additional data parts \( (U_{ij} - u, Z_t - z) \).

**Proofs of Theorem 2.1 and Corollary 2.1.** Theorem 2.1 and Corollary 2.1 follow directly from Lemma A.1 and from applying a central limit theorem for strictly stationary ergodic times series such as Theorem 9.5.5 in Karlin and Taylor (1975).

**APPENDIX B: DATA SOURCES**

Hourly spot prices of the German electricity market are provided by the European Energy Power Exchange (EPEX) (www.epexspot.com), hourly values of Germany’s gross electricity demand and electricity exchanges with other countries are provided by the European Network of Transmission System Operators for Electricity (www.entsoe.eu). German wind and solar power infeed data are provided by the transparency platform of the European Energy Exchange (www.eex-transparency.com). German air temperature data are available from the German Weather Service (www.dwd.de). The daily prices for natural gas are provided via the trading platform PEGAS, which is part of the European Energy Exchange (EEX) Group operated by Powernext (www.powernext.com). Daily prices for European CO2 Emission Allowances (EUA) and for the Amsterdam-Rotterdam-Antwerp (ARA) coal futures are provided via the websites of the EEX (www.eex.com).

**REFERENCES**


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