Online-Supplement to:

A Wavelet Method for Panel Models with Jump Discontinuities in the Parameters

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Abstract

This document contains additional theoretical discussions (Appendix A), the proofs of our theoretical results (Appendix B), and additional simulation results (Appendix C).

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A Further Discussions

In this section, we discuss the following three possible model extensions: first, the case in which endogeneity arises from an omitted factor structure, second, second, the case of dynamic panel data models, and, third, the case in which endogeneity is due to the presence of simultaneous equations.

A.1 Presence of Multifactor Errors

There is a growing literature on large panel models that allows for the presence of unobserved time-varying individual effects having an approximate factor structure such that

$$e_{it} = \Lambda_i^{'} F_t + \epsilon_{it},$$

where Λ_i is a $(d \times 1)$ vector of individual scores (or loadings) $\Lambda_i = (\Lambda_{i1}, \ldots, \Lambda_{id})'$ and F_t a $(d \times 1)$ vector of d common factors $F_{1t}, \ldots, F_{dt}$. Note that this extension provides a generalization of panel data models with additive effects and can be very useful in many application areas, especially when the unobserved individual effects are non-static over time; see, e.g., Pesaran (2006), Bai (2009), Ahn et al. (2013), Kneip et al. (2012), and Bada and Kneip (2014).

Leaving the factor structure in the error term and estimating the remaining parameters without explicitly considering the presence of a potential correlation between the observed regressors $X_{1,it}, \ldots, X_{P,it}$ and the unobserved effects Λ_i and F_t may lead to an endogeneity problem caused by these omitted model components. The problem with the presence of the factor structure in the error term is that such a structure can not be eliminated by differencing the observed variables or using a simple within-transformation. Owing to the potential correlation between the observable regressors $X_{1,it}, \ldots, X_{P,it}$ and the unobservable heterogeneity effects, we allow for the data generating process of $X_{p,it}$ to have the following rather general form:

$$X_{p,it} = \vartheta_{p,i}^{'} F_t + \Lambda_i^{'} G_{p,t} + a_p \Lambda_i^{'} F_t + \mu_{p,it}, \tag{1}$$

where $\vartheta_{p,i}$ is a $(d \times 1)$ vector of unknown individual scores, $G_{p,t}$ is a $(d \times 1)$ vector of unobservable common factors, a_p is a p-specific univariate coefficient, and μ_{it} is an individual specific term that is uncorrelated with ϵ_{it}, Λ_i, ϑ_i, F_t and G_t.

Rearranging (1), we can rewrite $X_{p,it}$ as

$$X_{p,it} = \vartheta_{p,i}^{*'} G_{p,t}^{*} + \mu_{p,it}, \tag{2}$$

where

$$\vartheta_{p,i}^{*'} = H(a_p \Lambda_i^{'} + \vartheta_{p,i}^{'}, \Lambda_i^{'}), \tag{3}$$

and

$$G_{p,t}^{*} = H^{-1}(F_t^{'}, G_{p,t}^{'})^{'}, \tag{4}$$

for some $(2d \times 2d)$ full rank matrix H. The role of H is only to ensure orthonormality and identify uniquely (up to a sign change) the elements of the factor structure so that $\sum_{t=1}^{T} G_{p,t}^{'*} G_{p,t}^{*}/T$ is the identity matrix and $\sum_{i=1}^{n} \vartheta_{p,i}^{*'} \vartheta_{p,i}^{*}/n$ is a diagonal matrix with ordered diagonal elements.

We can see from (1) that an ideal candidate for instrumenting $X_{p,it}$ is $\mu_{p,it}$. Since $\mu_{p,it}$ is unobserved, a feasible instrument can be obtained by

$$Z_{p,it} = X_{p,it} - \hat{\vartheta}_{p,i}^{*'} \hat{G}_{p,t}^{*}, \tag{5}$$

where $\hat{G}^{*\prime}_{p,t}$ is the t-th row element of the $(2d \times 1)$ matrix containing the eigenvectors corresponding to the ordered eigenvalues of the covariance matrix of $X_{p,it}$ and $\hat{\vartheta}^{*\prime}_{p,i}$ is the projection of $\hat{G}^{*\prime}_{p,t}$ on $X_{p,it}$. If d is unknown, one can estimate the dimension of $\vartheta^{*\prime}_{p,i}G^{*}_{p,t}$ by using an appropriate panel information criterion; see, e.g., Bai and Ng (2002) and Onatski (2010). A crucial assumption about the form of dependency in $\mu_{p,it}$ is that, for all T and n, and every $i \leq n$ and $t \leq T$,

1. $\sum_{s=1}^{T} |E(\mu_{p,it}\mu_{p,is})| \leq M$ and
2. $\sum_{k=1}^{n} |E(\mu_{p,it}\mu_{p,kt})| \leq M$.

Bai (2003) proves the consistency of the principal component estimator when additionally $\frac{1}{T}\sum_{t=1}^{T} G^{*\prime}_{p,t}G^{*}_{p,t} \xrightarrow{p} \Sigma_{G^*_p}$ for some $(2d \times 2d)$ positive definite matrix $\Sigma_{G^*_p}$, $||\vartheta^{*}_{p,i}|| \leq M$ for all i and p, and $||\frac{1}{n}\sum_{i=1}^{n} \vartheta^{*\prime}_{p,i}\vartheta^{*}_{p,i} - \Sigma_{\vartheta^*_p}|| \longrightarrow 0$, as $n \to \infty$ for some $(2d \times 2d)$ positive definite matrix $\Sigma_{\vartheta^*_p}$.

By instrumenting $X_{p,it}$ with $Z_{p,it}$ in (5), we can consistently estimate the jumping slope parameters as before.

A.2 Dynamic Panel Data Models

Generally, finding suitable instrumental variables that satisfy the necessary IV assumptions can be challenging. However, in the case of a dynamic model where the lagged dependent variable, $Y_{i,t-1}$, belongs to the set of regressor variables, we can easily find several natural candidates which could be inspired from the structure of the model itself. For instance, let us consider the simple case where the lagged dependent variable, $Y_{i,t-1}$, is the only regressor variable and where the errors, e_{it}, are serially uncorrelated innovations. Taking the first difference of the dynamic model,

$$Y_{it} = \alpha_i + Y_{it-1}\beta_t + e_{it}, \tag{6}$$

eliminates the unknown individual effects and transforms the regression equation (6) to

$$\Delta Y_{it} = Y_{it-1}\beta_t - Y_{it-2}\beta_{t-1} + \Delta e_{it}. \tag{7}$$

It is easily seen that Y_{it-1} on the right hand side is correlated with the new error $\Delta e_{it} = (e_{it} - e_{it-1})$ since Y_{it-1} contains e_{it-1}. A natural instrumental variable for this endogeneous regressor could be for instance Y_{it-2} because it is not correlated with Δe_{it} and, by construction, correlated with Y_{it-1}. Alternatively candidates such as Y_{it-3} or ΔY_{it-2} also satisfy the IV requirements.

A.3 Two-Step SAW for Jump Reverse Causality

Besides the issues of omitted variables and dynamic dependent variables, another important source of endogeneity is the phenomenon of reverse causality. This occurs when the data, e.g., is generated by a system of simultaneous equations.

Consider the following two-equation simultaneous equation system:

$$Y_{it} = \mu + \sum_{p=1}^{P} X_{p,it}\beta_{t,p} + \alpha_i + \theta_t + e_{it}, \tag{8}$$

and

$$X_{q,it} = b_t Y_{it} + \sum_{p\in\{1,\dots,P\}\setminus\{q\}} X_{p,it} d_{t,p} + v + u_i + \vartheta_t + \nu_{it}, \tag{9}$$

for some a $q \in \{1,\dots,P\}$, where $b_t \neq 1/\beta_{t,q}$, and the parameters v, u_i, and ϑ_t are unknown parameters.

Neglecting the structural form of $X_{q,it}$ in Equation (9) and estimating the regression function (8) without instrumenting this variable results in an inconsistent estimation since $X_{q,it}$ and e_{it} are correlated (due to the presence of Y_{it} in Equation (9)). A natural way to overcome this type of endogeneity problem is to use the fitted variable obtained from Equation (9) as an instrument after replacing Y_{it} with its expression in (8). However, our model involves an additional complication related to the time-changing character of $\beta_{t,q}$ and the presence of the unobservable heterogeneity effects that render such two-stage least squares estimators problematic. Inserting (8) in (9) and rearranging it leads to a panel model with time-varying unobservable individual effects:

$$X_{q,it} = \sum_{p\in\{1,\dots,P\}\setminus\{q\}} X_{p,it} d^*_{t,p} + \vartheta^*_{1t} + u_i\vartheta^*_{2t} + \alpha_i\vartheta^*_{3t} + \varepsilon_{it}, \tag{10}$$

where

$$\begin{aligned} d^*_{t,p} &= b_t\beta_{t,p} + d_{t,p}, \\ \vartheta^*_{1t} &= \tfrac{b_t\mu + b_t\theta_t + \vartheta_t + v}{1 - b_t\beta_{t,p}}, \\ \vartheta^*_{2t} &= \tfrac{1}{1 - b_t\beta_{t,p}}, \\ \vartheta^*_{3t} &= \tfrac{b_t\mu + b_t\theta_t + \vartheta_t + v}{1 - b_t\beta_{t,p}}, \text{ and} \\ \varepsilon_{it} &= b_t e_{it} + \varepsilon_{it}. \end{aligned}$$

Note that the regression model in (10) can be considered a special case of the model with multifactor errors discussed above. A potential instrument for $X_{q,it}$ in (8) is then

$$Z_{q,it} = \sum_{p\in\{1,\dots,P\}\setminus\{q\}} X_{p,it} \hat{d}^*_{t,p} + \hat{\vartheta}'_i \hat{G}_t, \tag{11}$$

where $\hat{d}^*_{t,p}$ and $\hat{\vartheta}'_i\hat{G}_t$ are the estimators of b_t and $\vartheta'_i G_t = \vartheta^*_{1t} + u_i\vartheta^*_{2t} + \alpha_i\vartheta^*_{3t}$, respectively, and which can be obtained from (10) by using the instruments proposed above to control for the omitted factor structure $\vartheta^*_{1t} + u_i\vartheta^*_{2t} + \alpha_i\vartheta^*_{3t}$.

B Theoretical Results and Proofs

B.1 Proofs of Section 2

To prove Proposition 1, we need following Lemma.

Lemma 1 (A) *Let $T^* = 2^{L-1}$ for some integer $L \geq 2$ and $\gamma^* = (\gamma^{*'}_1, \dots, \gamma^{*'}_T)'$ a $(T^* \times P^*)$ matrix that possesses exactly one jump at $\tau \in \{1,\dots,T^*\}$ in one of its P^* column vectors such that*

$$\gamma^{*'}_t = \begin{cases} \gamma^{*'}_\tau & \text{for} \quad t \in \{1,\dots,\tau\} \\ \gamma^{*'}_{\tau+1} \neq \beta_\tau & \text{for} \quad t \in \{\tau+1,\dots,T\}. \end{cases}$$

Let $W^*_{l,k}(t)$ be defined as follows

$$W^*_{l,k}(t) = \begin{cases} A^*_{1,1} & \text{if} \quad l = 1, \quad \text{and} \\ A^*_{l,2k-1} I_{l,2k-1}(t) - A^*_{l,2k} I_{l,2k}(t) & \text{if} \quad l > 1, \end{cases} \tag{12}$$

where $A^*_{l,m}$ *are arbitrary invertible matrices and* $I_{l,m}(t)$ *is the indicator function that carries the value one if* $t \in \{2^{L-l}(m-1)+1, \ldots, 2^{L-l}m\}$ *and zero otherwise. There then exists at most* L *non-zero coefficient vectors such that*

$$\gamma_t^* = \sum_{l=1}^{l_\tau} W^*_{lk_l}(t)\underline{b}^*_{lk_l},$$

where $\underline{b}^*_{lm}$ *are* $(P^* \times 1)$ *vectors,* $l_\tau \leq L$ *and* $k_l \leq 2^{l-2}$.

Proof of Lemma 1 (A): To prove the Lemma, we show that γ_t^* can be reconstructed by using a number of (multivariate) wavelet basis that is smaller than or equal to L.

Due to the construction of the indicator function $I_{l,m}(t)$, we have at $\tau \in \{1, \ldots, T^*\}$ a unique index tuple, say (l_τ, k_{l_τ}), where $l_\tau \in \{2, \ldots, L\}$ and $k_{l_\tau} \in \{1, \ldots, 2^{l_\tau - 2}\}$, such that

$$W^*_{l_\tau k_{l_\tau}}(\tau) = A^*_{l_\tau, 2k_{l_\tau}-1} \quad \text{and} \quad W^*_{l_\tau k_{l_\tau}}(\tau+1) = -A^*_{l_\tau, 2k_{l_\tau}}.$$

Now, define the time interval $\mathcal{I}_l$, for each $l = 1, \ldots, l_\tau$, as follows:

$$\mathcal{I}_l = \{t \in \{1, \ldots, T^*\} | W^*_{l,k_l}(t) \neq 0\}.$$

Due to the construction of the wavelet basis $W^*_{l,k_l}(t)$, derived by the indicator function $I_{l,k}(t)$, we can verify that

$$\bigcup_{l=1}^{l_\tau} \mathcal{I}_l = \{1, \ldots, T^*\}$$

and

$$\mathcal{I}_{l_\tau} \subset \mathcal{I}_{l_\tau - 1} \subset \cdots \subset \mathcal{I}_2 \subseteq \mathcal{I}_1 = \{1, \ldots, T^*\}.$$

Starting with the thinnest interval $\mathcal{I}_{l_\tau}$ that contains the jump location, we first define the sparse vector

$$\gamma_t^{*(l_\tau)} = \begin{cases} \gamma^*_\tau & \text{if} \quad t \in \mathcal{I}_{l_\tau} \cap \{t | t \leq \tau\} \\ \gamma^*_{\tau+1} & \text{if} \quad t \in \mathcal{I}_{l_\tau} \cap \{t | t > \tau\} \\ 0 & \text{otherwise.} \end{cases}$$

Because $\gamma^*_{\tau+1} \neq \gamma^*_\tau$, we can uniquely determine a non-zero coefficient vector $\underline{b}_{l_\tau, k_{l_\tau}}$ and an interval constant vector $\gamma^{*(l_\tau)} \neq \{\gamma^*_\tau, \gamma^*_{\tau+1}\}$ for any arbitrary invertible matrices $A^*_{l_\tau, 2k_{l_\tau}-1}$ and $A^*_{l_\tau, 2k_{l_\tau}}$ such that

$$\gamma_t^{*(l_\tau)} = \begin{cases} \gamma^{*(l_\tau)} + A^*_{l_\tau, 2k_{l_\tau}-1}\underline{b}_{l_\tau, k_{l_\tau}} & \text{if} \quad t \in \mathcal{I}_{l_\tau} \cap \{t | t \leq \tau\} \\ \gamma^{*(l_\tau)} - A^*_{l_\tau, 2k_{l_\tau}}\underline{b}_{l_\tau, k_{l_\tau}} & \text{if} \quad t \in \mathcal{I}_{l_\tau} \cap \{t | t > \tau\} \\ 0 & \text{else.} \end{cases} \tag{13}$$

Since $W^*_{l_\tau k_{l_\tau}}(\tau) = A^*_{l_\tau, 2k_{l_\tau}-1}$ and $W^*_{l_\tau k_{l_\tau}}(\tau+1) = -A^*_{l_\tau, 2k_{l_\tau}}$ at the jump location, we can rewrite (13) as

$$\gamma_t^{*(l_\tau)} = \begin{cases} \gamma^{*(l_\tau)} + W^*_{l_\tau k_{l_\tau}}(\tau)\underline{b}_{l_\tau, k_{l_\tau}} & \text{if} \quad t \in \mathcal{I}_{l_\tau} \\ 0 & \text{else.} \end{cases} \tag{14}$$

Now, we proceed with the second thinnest interval $\mathcal{I}_{l_\tau-1}$ and define iteratively a second sparse vector with one jump at $\mathcal{I}_{l_\tau-1}$. Let

$$\gamma_t^{*(l_\tau-1)} = \begin{cases} \gamma_t^* & \text{if} \quad t \in \mathcal{I}_{l_\tau-1} \setminus \mathcal{I}_{l_\tau} \\ \gamma^{*(l_\tau)} & \text{if} \quad t \in \mathcal{I}_{l_\tau} \\ 0 & \text{else.} \end{cases}$$

Note that γ_t^* is constant over $\mathcal{I}_{l_\tau-1} \setminus \mathcal{I}_{l_\tau}$; it can be either γ_τ^* or $\gamma_{\tau+1}^*$. Now, because $\gamma^{*(l_\tau)} \neq \{\gamma_\tau^*, \gamma_{\tau+1}^*\}$, we can determine a second unique non-zero coefficient $\underline{b}_{l_\tau-1,k_{l_\tau-1}}$ and a second unique constant $\gamma^{*(l_\tau-1)} \neq \{\gamma_\tau^*, \gamma_{\tau+1}^*, \gamma^{*(l_\tau)}\}$ for any arbitrary invertible matrices $A^*_{l_\tau-1,2k_{l_\tau-1}-1}$ and $A^*_{l_\tau-1,2k_{l_\tau-1}}$ such that

$$\gamma_t^{*(l_\tau-1)} = \begin{cases} \gamma^{*(l_\tau-1)} + A^*_{l_\tau-1,2k_{l_\tau-1}-1}\underline{b}_{l_\tau-1,k_{l_\tau-1}} & \text{if} \quad t \in \mathcal{I}_{l_\tau-1} \setminus \mathcal{I}_{l_\tau} \\ \gamma^{*(l_\tau-1)} + A^*_{l_\tau-1,2k_{l_\tau-1}}\underline{b}_{l_\tau-1,k_{l_\tau-1}} & \text{if} \quad t \in \mathcal{I}_{l_\tau} \\ 0 & \text{else} \end{cases}$$

and, because of the same argument as above,

$$\gamma_t^{*(l_\tau-1)} = \begin{cases} \gamma^{*(l_\tau-1)} + W^*_{l_\tau-1,k_{l_\tau-1}}\underline{b}_{l_\tau-1,k_{l_\tau-1}} & \text{if} \quad t \in \mathcal{I}_{l_\tau-1} \\ 0 & \text{else.} \end{cases}$$

Recall that $W^*_{l_\tau,k_{l_\tau}}(t) = 0$ for all $t \notin \mathcal{I}_{l_\tau}$, adding $W^*_{l_\tau,k_{l_\tau}}(t)\underline{b}_{l,k_l}$ on both sides, gives

$$\gamma_t^{*(l_\tau-1)} + W^*_{l_\tau,k_{l_\tau}}(t)\underline{b}_{l,k_l} = \begin{cases} \gamma_t^* + W^*_{l_\tau,k_{l_\tau}}(t)b_{l,k_l} & \text{if} \quad t \in \mathcal{I}_{l_\tau-1} \setminus \mathcal{I}_{l_\tau} \\ \gamma^{*(l_\tau)} + W^*_{l_\tau,k_{l_\tau}}(t)b_{l,k_l} & \text{if} \quad t \in \mathcal{I}_{l_\tau} \\ 0 & \text{else.} \end{cases}$$

Because $\gamma^{*(l_\tau)} + W^*_{l_\tau,k_{l_\tau}}(t)b_{l,k_l} = \gamma_t^*$ for all $t \in \mathcal{I}_{l_\tau}$, we can verify

$$\gamma_t^{*(l_\tau-1)} + W^*_{l_\tau,k_{l_\tau}}(t)b_{l,k_l} = \begin{cases} \gamma^{*(l_\tau-1)} + \sum_{l=l_\tau-1}^{l_\tau} W^*_{l,k_l}(t)b_{l,k_l} = \gamma_t^* & \text{if} \quad t \in \mathcal{I}_{l_\tau-1} \\ 0 & \text{else.} \end{cases}$$

Replacing $\gamma_t^{*(t_\tau-1)}$ with $\gamma*_t^{(t_\tau-2)}$ and proceeding recursively with one jump sparse vector constructions over $\gamma_t^{*(l_\tau-l)}$, for $l \in \{2, \ldots, l_\tau\}$, we end up with

$$\gamma_t^* = \gamma^{*(1)} + \sum_{l=2}^{l_\tau} W^*_{l,k_l}(t)\underline{b}_{l,k_l} \qquad \forall t \in \{1, \ldots, T^*\}.$$

Finally, since $\gamma^{*(1)}$ is a constant and $W^*_{1,1}(t) = A^*_{1,1}$, $\forall t \in \{1, \ldots, T\}$, we can express γ_t^* in terms of $l_\tau \leq L$ wavelet basis and l_τ non-zero coefficient vectors for any arbitrary invertible matrices $A^*_{l,m}$ such that

$$\gamma_t^* = \sum_{l=1}^{l_\tau} W^*_{l,k_l}(t)\underline{b}_{l,k_l} \qquad \forall t \in \{1, \ldots, T^*\}.$$

This completes the proof. $\square$

Proposition 1 basically generalizes Lemma 1 to the case of $(T^* \times P^*)$ matrices with multi-jumps.

Proof of Proposition 1: To prove the preposition, we first define $T^* = T - 1$, $\gamma_t^* = \gamma_{t+1}$, and $W_{l,k}^*(t) = W_{l,k}(t+1)$, for $t = 1, \ldots, T^*$. The basic idea of the proof is to expand the original $(T^* \times \underline{P})$ matrix $\gamma^* = (\gamma_1^{*'}, \ldots, \gamma_{T^*}^{*'})^*$ in a series of S matrices so that each new designed matrix contains at most one jump, and then make use of Lemma 1 (A).

The row presentation of γ^* can be expressed as follows

$$\gamma^* = \begin{pmatrix} \gamma_2 \\ \vdots \\ \gamma_{\tau_1} \\ \gamma_{\tau_1+1} \\ \vdots \\ \gamma_{\tau_2} \\ \gamma_{\tau_2+1} \\ \vdots \\ \gamma_{\tau_S+1} \\ \vdots \\ \gamma_T \end{pmatrix} = \begin{pmatrix} \gamma_{\tau_1} \\ \vdots \\ \gamma_{\tau_1} \\ \gamma_{\tau_2} \\ \vdots \\ \gamma_{\tau_2} \\ \gamma_{\tau_3} \\ \vdots \\ \gamma_{\tau_{S+1}} \\ \vdots \\ \gamma_{\tau_{S+1}} \end{pmatrix}$$

for $\{\tau_1, \ldots, \tau_S\} \subset \{2, \ldots, T\}$. We can transform γ^* in a series of $S+1$ matrices, $\overline{\gamma}_{\tau_{(1)}}, \ldots, \overline{\gamma}_{\tau_{(S)}}$ as follows:

$$\underbrace{\begin{pmatrix} \gamma_{\tau_1} \\ \vdots \\ \gamma_{\tau_1} \\ \gamma_{\tau_2} \\ \vdots \\ \gamma_{\tau_2} \\ \gamma_{\tau_3} \\ \vdots \\ \gamma_{\tau_S} \\ \gamma_{\tau_{S+1}} \\ \vdots \\ \gamma_{\tau_{S+1}} \end{pmatrix}}_{\gamma} = \underbrace{\begin{pmatrix} \gamma_{\tau_1} - \gamma_{\tau_2} \\ \vdots \\ \gamma_{\tau_1} - \gamma_{\tau_2} \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\overline{\gamma}_{\tau_{(1)}}} + \ldots + \underbrace{\begin{pmatrix} \gamma_{\tau_{S-1}} - \gamma_{\tau_S} \\ \vdots \\ \gamma_{\tau_{S-1}} - \gamma_{\tau_S} \\ \gamma_{\tau_{S-1}} - \gamma_{\tau_S} \\ \vdots \\ \gamma_{\tau_{S-1}} - \gamma_{\tau_S} \\ \gamma_{\tau_{S-1}} - \gamma_{\tau_S} \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\overline{\gamma}_{\tau_{(S)}}} + \underbrace{\begin{pmatrix} \gamma_{\tau_S} \\ \vdots \\ \gamma_{\tau_S} \\ \gamma_{\tau_S} \\ \vdots \\ \gamma_{\tau_S} \\ \gamma_{\tau_S} \\ \vdots \\ \gamma_{\tau_S} \\ \gamma_{\tau_{S+1}} \\ \vdots \\ \gamma_{\tau_{S+1}} \end{pmatrix}}_{\overline{\gamma}_{(\tau_S+1)}},$$

so that each new $(T^* \times \underline{P})$ matrix processes exactly one jump (except $\overline{\gamma}_{\tau_S+1}$, which is constant over time). From Proposition 1, we know that each vector of the matrix $\overline{\gamma}_{\tau_s} = (\overline{\gamma}_{1,\tau_s}^{'}, \ldots, \overline{\gamma}_{T^*,\tau_s}^{'})'$, $s = 1, \ldots, S$, has a unique expansion of the form

$$\overline{\gamma}_{t,\tau_{(s)}} = \sum_{l=1}^{L} \sum_{k=1}^{K_l} W_{lk}^*(t) \underline{b}_{lk}^{(s)} \qquad \forall t \in \{1, \ldots, T^*\}$$

with at most L non-zero coefficient vectors. The fact that $\gamma_t^* = \sum_{s=1}^{S+1} \overline{\gamma}_{t,\tau_s}$ completes the proof. $\square$

B.2 Proofs of Section 2

Proof of Lemma 1: The IV estimator of our (modified) wavelets coefficients is given by

$$\begin{aligned}
\tilde{b}_{l,k,p} &= \frac{1}{n(T-1)} \sum_{i=1}^{n} \sum_{t=2}^{T} \mathcal{Z}_{lk,it,p} \Delta y_{it}, \\
&= \frac{1}{n(T-1)} \sum_{i=1}^{n} \sum_{t=2}^{T} \mathcal{Z}_{lk,it,p} (\sum_{l=1}^{L} \sum_{k=1}^{K_l} \sum_{q=1}^{P} \mathcal{Z}_{lk,it,q} b_{l,k,q} + \Delta e_{it}), \\
&= b_{l,k,p} + \frac{1}{n(T-1)} \sum_{i=1}^{n} \sum_{t=2}^{T} \sum_{q=1}^{P} \mathcal{Z}_{lk,it,p} \Delta e_{it}.
\end{aligned}$$

The last equality is due to the orthonormality conditions (A) and (B). Subtracting $b_{l,k,p}$ from both sides and multiplying by $\sqrt{n(T-1)}$, we get, for $l > 1$,

$$\begin{aligned}
\sqrt{n(T-1)}(\tilde{b}_{l,k,p} - b_{l,k,p}) &= \frac{1}{\sqrt{n(T-1)}} \sum_{i=1}^{n} \sum_{t=2}^{T} \sum_{q=1}^{P} \mathcal{Z}_{lk,it,q} \Delta e_{it}, \\
&= \frac{1}{\sqrt{n(T-1)}} \sum_{i=1}^{n} \sum_{t=2}^{T} \sum_{q=1}^{P} W_{lk,pq}(t) Z_{it,q} \Delta e_{it}, \\
&= \frac{1}{\sqrt{n(T-1)}} \sum_{i=1}^{n} \sum_{t=2}^{T} \sum_{q=1}^{P} A_{l,2k,pq} H_{l,2k}(t) Z_{it,q} \Delta e_{it} \\
&- \frac{1}{\sqrt{n(T-1)}} \sum_{i=1}^{n} \sum_{t=2}^{T} \sum_{q=1}^{P} A_{l,2k-1,pq} H_{l,2k-1}(t) Z_{it,q} \Delta e_{it}, \\
&= \frac{1}{\sqrt{n(2^{L-l-1}-1)}} \sum_{q=1}^{P} A_{l,2k,pq} \sum_{i=1}^{n} \sum_{t \in \{H_{l,2k}(t) \neq 0\}} Z_{it,q} \Delta e_{it} \\
&- \frac{1}{\sqrt{n(2^{L-l-1}-1)}} \sum_{q=1}^{P} A_{l,2k-1,pq} \sum_{i=1}^{n} \sum_{t \in \{H_{l,2k-1}(t) \neq 0\}} Z_{it,q} \Delta e_{it},
\end{aligned}$$

where $W_{lk,pq}(t)$ and $A_{l,m,pq}$ are the (p,q)- elements of the matrices $W_{l,k}(t)$ and $A_{l,m}$, respectively. and, for $l = 1$,

$$\sqrt{n(T-1)}(\tilde{b}_{1,1,p} - b_{1,1,p}) = \frac{1}{\sqrt{n(2^L-1)}} \sum_{q=1}^{P} A_{1,1,pq} \sum_{i=1}^{n} \sum_{t=2}^{T} Z_{it,q} \Delta e_{it}.$$

From the eigenvalue assumption in Assumption B (i) it follows that with probability 1 $||A_{l,2k}||_2^2$ and $||A_{l,2k-1}||_2^2$ are bounded uniformly in l, k. Therefore, there exists a constant $\mathcal{C} < \infty$ such that with probability 1

$$\sqrt{n(T-1)}|\tilde{b}_{l,k,p} - b_{l,k,p}| \tag{15}$$
$$\leq \mathcal{C} \cdot \frac{\underline{P}}{\sqrt{n(2^{L-l-1}-1)}} \max_q \{|\sum_{i=1}^n \sum_{t\in\{H_{l,2k}(t)\neq 0\}} Z_{it,q}\Delta e_{it}|, |\sum_{i=1}^n \sum_{t\in\{H_{l,2k-1}(t)\neq 0\}} Z_{it,q}\Delta e_{it}|\}$$

and

$$\sqrt{n(T-1)}|\tilde{b}_{1,1,p} - b_{1,1,p}| \leq \mathcal{C} \cdot \frac{P}{\sqrt{n(2^L-1)}} \max_q \{|\sum_{i=1}^n \sum_{t=2}^T Z_{it,q}\Delta e_{it}|\} \tag{16}$$

For arbitrary $d \geq 0$ Assumption C implies that for all $1 \leq s < s' \leq T$ and any $p = 1, \dots, P$

$$P\left(\frac{1}{\sqrt{n(s'-s+1)}}|\sum_{i=1}^n \sum_{t=s+1}^{s'} Z_{it,p}\Delta e_{it}| \geq \left(\frac{(\sqrt{2}+d)\log(T-1)}{\delta_1}\right)^{1/\delta_2} \cdot \sqrt{M}\right) \tag{17}$$
$$\leq \delta_0 \exp(-(\sqrt{2}+d)\log(T-1)) = \delta_0(T-1)^{-(\sqrt{2}+d)}$$

By (15) - (17) we can conclude that with $\mathcal{M}_d = 2\mathcal{C}P\sqrt{M}(\frac{\sqrt{2}+d}{\delta_1})^{1/\delta_2}$

$$P\left(\sup_{l,k,p} \sqrt{n(T-1)}\left|\tilde{b}_{l,k,p} - b_{l,k,p}\right| > \mathcal{M}_d \log(T-1)^{1/\delta_2}\right)$$
$$\leq \sum_{l,k,p} P\left(\sqrt{n(T-1)}\left|\tilde{b}_{l,k,p} - b_{l,k,p}\right| > \mathcal{M}_d \log(T-1)^{1/\delta_2}\right),$$
$$\leq 2^{L-1}\underline{P}\delta_0(T-1)^{-(\sqrt{2}+d)} = \delta_0\underline{P}(T-1)^{-(\sqrt{2}+d)+1},$$

where $\sum_{l,k,p}$ denotes the triple summation $\sum_{l=1}^L \sum_{k=1}^{K_l} \sum_{p=1}^{\underline{P}}$. Assertion (i) of the lemma is an immediate consequence, while (ii) follows with $\mathcal{M} := \mathcal{M}_0$. $\Box$

Proof of Theorem 1: We have first to prove that (i) : $\sup_t |\tilde{\gamma}_{t,p} - \gamma_{t,p}| = o_p(1)$ for all $p \in \{1, \dots, \underline{P}\}$ if $\sqrt{T-1}\lambda_{n,T} \to 0$, as $n, T \to \infty$ or $n \to \infty$ and T is fixed, and then conclude that (ii) : $\frac{1}{T-1}\sum_{t=2}^T ||\tilde{\gamma}_t - \gamma_t||^2 = O_p((\log(T-1)/n)^\kappa)$, if $\sqrt{T-1}\lambda_{n,T} \sim (\log(T-1)/n)^{\kappa/2}$, for $\kappa \in]0,1]$.

By construction,

$$\tilde{\gamma}_{t,p} - \gamma_{t,p} = \sum_{q=1}^{\underline{P}} \sum_{l=1}^L \sum_{k=1}^{K_l} W_{lk,pq}(t)\hat{b}_{l,k,q} - \sum_{q=1}^{\underline{P}} \sum_{l=1}^L \sum_{k=1}^{K_l} W_{lk,pq}(t) b_{l,k,q}, \tag{18}$$

where

$$\hat{b}_{l,k,q} = \tilde{b}_{l,k,q} - \tilde{b}_{l,k,q}(|\tilde{b}_{l,k,q}| < \lambda_{n,T}). \tag{19}$$

and

$$W_{lk,pq}(t) = A_{l,2k,pq}(t)H_{l,2k}(t) - A_{l,2k-1,pq}(t)H_{l,2k-1}(t), \tag{20}$$
$$= \sqrt{2^{l-2}}A_{l,2k,pq}\mathbf{I}(H_{l,2k}(t) \neq 0) - \sqrt{2^{l-2}}A_{l,2k-1,pq}\mathbf{I}(H_{l,2k-1}(t) \neq 0).$$

Plugging (19) and (20) in (18) and using the absolute value inequality, we get

$$
\begin{aligned}
|\tilde{\gamma}_{t,p}-\gamma_{t,p}| &\le \sum_{q=1}^{\underline{P}}\sum_{l=1}^{L}\sum_{k=1}^{K_l}\sqrt{2^{l-2}}|A_{l,2k,pq}\mathbf{I}(H_{l,2k}(t)\neq 0)(\tilde{b}_{l,k,q}-b_{l,k,q})| \\
&+\sum_{q=1}^{\underline{P}}\sum_{l=1}^{L}\sum_{k=1}^{K_l}\sqrt{2^{l-2}}|A_{l,2k,pq}\mathbf{I}(H_{l,2k}(t)\neq 0)\tilde{b}_{l,k,q}\mathbf{I}(|\tilde{b}_{l,k,q}|<\lambda_{n,T})| \\
&+\sum_{q=1}^{\underline{P}}\sum_{l=1}^{L}\sum_{k=1}^{K_l}\sqrt{2^{l-2}}|A_{l,2k-1,pq}\mathbf{I}(H_{l,2k-1}(t)\neq 0)(\underline{\tilde{b}}_{l,k,q}-\underline{b}_{l,k,q})| \\
&+\sum_{q=1}^{\underline{P}}\sum_{l=1}^{L}\sum_{k=1}^{K_l}\sqrt{2^{l-2}}|A_{l,2k,pq}\mathbf{I}(H_{l,2k-1}(t)\neq 0)\tilde{b}_{l,k,q}\mathbf{I}(|\tilde{b}_{l,k,q}|<\lambda_{n,T})|, \\
&= a+b+c+d.
\end{aligned}
$$

Because $\tilde{b}_{l,k,p}\mathbf{I}(|\tilde{b}_{l,k,p}|<\lambda_{n,T})<\lambda_{n,T}$ and $|\tilde{b}_{l,k,p}-b_{l,k,p}|\le \sup_{l,k,p}|\tilde{b}_{l,k,p}-b_{l,k,p}|$ for all $p\in\{1,\dots,\underline{P}\}$, we can write

$$
\begin{aligned}
a &\le \sup_{l,k,p}|\tilde{b}_{l,k,p}-b_{l,k,p}|\sum_{q=1}^{\underline{P}}\sum_{l=1}^{L}\sum_{k=1}^{K_l}|A_{l,2k,pq}\sqrt{2^{l-2}}\mathbf{I}(H_{l,2k}(t)\neq 0)|,\\
b &\le \lambda_{n,T}\sum_{q=1}^{\underline{P}}\sum_{l=1}^{L}\sum_{k=1}^{K_l}|A_{l,2k,pq}\sqrt{2^{l-2}}\mathbf{I}(H_{l,2k}(t)\neq 0)|,\\
c &\le \sup_{l,k,p}|\tilde{b}_{l,k,p}-b_{l,k,p}|\sum_{q=1}^{\underline{P}}\sum_{l=1}^{L}\sum_{k=1}^{K_l}|A_{l,2k-1,pq}\sqrt{2^{l-2}}\mathbf{I}(H_{l,2k}(t)\neq 0)|,\ \text{and}\\
d &\le \lambda_{n,T}\sum_{q=1}^{\underline{P}}\sum_{l=1}^{L}\sum_{k=1}^{K_l}|A_{l,2k-1,pq}\sqrt{2^{l-2}}\mathbf{I}(H_{l,2k-1}(t)\neq 0)|.
\end{aligned}
$$

From the eigenvalue assumption in Assumption B (i) it follows that $E(||A_{l,2k}||^4)$ and $E(||A_{l,2k-1}||^4)$ are bounded uniformly in l and k. We can deduce that

$$
\sum_{q=1}^{\underline{P}}\sum_{l=1}^{L}\sum_{k=1}^{K_l}|A_{l,2k,pq}\sqrt{2^{l-2}}\mathbf{I}(H_{l,2k}(t)\neq 0)| = O_p(1)\sum_{l=1}^{L}\sum_{k=1}^{K_l}|\sqrt{2^{l-2}}\mathbf{I}(H_{l,2k}(t)\neq 0)|\ \text{ and}
$$

$$
\sum_{q=1}^{\underline{P}}\sum_{l=1}^{L}\sum_{k=1}^{K_l}|A_{l,2k-1,pq}\sqrt{2^{l-2}}\mathbf{I}(H_{l,2k-1}(t)\neq 0)| = O_p(1)\sum_{l=1}^{L}\sum_{k=1}^{K_l}|\sqrt{2^{l-2}}\mathbf{I}(H_{l,2k-1}(t)\neq 0)|.
$$

Moreover, from the construction of $H_{l,2k}(t)$ and $H_{l,2k-1}(t)$, we can easily verify that

$$
\sup_t\sum_{l=1}^{L}\sum_{k=1}^{K_l}\sqrt{2^{l-2}}\mathbf{I}(H_{l,2k-1}(t)\neq 0)=\sum_{l=1}^{L}\sqrt{2^{l-2}}=O(\sqrt{2^{L-1}})=O(\sqrt{T-1})
$$

By Lemma 1, we can infer that

$$
\begin{aligned}
\sup_{t,p}|\tilde{\gamma}_{t,p}-\gamma_{t,p}| &= \sup_{l,k,p}|\tilde{b}_{l,k,p}-b_{l,k,p}|\times O_p(\sqrt{T-1})+\lambda_{n,T}\times O_p(\sqrt{T-1}),\\
&= O_p\big(\sqrt{\tfrac{\log(T-1)}{n}}+\sqrt{T-1}\lambda_{n,T}\big). \qquad (21)
\end{aligned}
$$

Assertion (i) follows immediately if $\sqrt{T-1}\lambda_{n,T} \to 0$ with $\log(T-1)/n \to 0$, as $n, T \to \infty$.

Consider Assertion (ii). Let $\mathcal{L}_p := \{(l,k)|b_{l,k,p} = 0\}$ denote the set of double indexes corresponding to the non-zero true wavelet coefficients so that $\gamma_{t,p} = \sum_{q=1}^{P}\sum_{l=1}^{L}\sum_{k=1}^{K_l} W_{l,k,pq}(t)b_{l,k,q}$ can be written as

$$\gamma_{t,p} = \sum_{q=1}^{P}\sum_{(l,k)\in\mathcal{L}_p} W_{l,k,pq}(t)b_{l,k,q},$$

and $\tilde{\gamma}_{t,p} = \sum_{q=1}^{P}\sum_{l=1}^{L}\sum_{k=1}^{K_l} W_{lk,pq}(t)\hat{b}_{l,k,q}$ as

$$\tilde{\gamma}_{t,p} = \sum_{q=1}^{P}\sum_{(l,k)\in\mathcal{L}_p} W_{lk,pq}(t)\hat{b}_{l,k,q} + \sum_{q=1}^{P}\sum_{(l,k)\notin\mathcal{L}_p} W_{lk,pq}(t)\hat{b}_{l,k,q}.$$

The difference, can be written as

$$\tilde{\gamma}_{t,p} - \gamma_{t,p} = \sum_{q=1}^{P}\sum_{(l,k)\in\mathcal{L}_p} W_{lk,pq}(t)(\hat{b}_{l,k,q} - b_{l,k,q}) + \sum_{q=1}^{P}\sum_{(l,k)\notin\mathcal{L}_p} W_{lk,pq}(t)\hat{b}_{l,k,q}.$$

Averaging the square, we get

$$\begin{aligned}
\frac{1}{T-1}\sum_{t=2}^{T-1}(\tilde{\gamma}_{t,p} - \gamma_{t,p})^2 &= \frac{1}{T-1}\sum_{t=2}^{T-1}\left(\sum_{q=1}^{P}\sum_{(l,k)\in\mathcal{L}_p} W_{lk,pq}(t)(\hat{b}_{l,k,q} - b_{l,k,q})\right)^2 \\
&+ \frac{1}{T-1}\sum_{t=2}^{T-1}\left(\sum_{q=1}^{P}\sum_{(l,k)\notin\mathcal{L}_p} W_{lk,pq}(t)\hat{b}_{l,k,q}\right)^2 \\
&- \frac{1}{T-1}\sum_{t=2}^{T-1}\left(\sum_{q=1}^{P}\sum_{(l,k)\in\mathcal{L}_p} W_{lk,pq}(t)(\hat{b}_{l,k,q} - b_{l,k,q})\right)\times \\
&\left(\sum_{q=1}^{P}\sum_{(l,k)\notin\mathcal{L}_p} W_{lk,pq}(t)\hat{b}_{l,k,q}\right), \\
&= \frac{1}{T-1}\sum_{t=2}^{T-1} e_t^2 + \frac{1}{T-1}\sum_{t=2}^{T-1} f_t^2 - \frac{1}{T-1}\sum_{t=2}^{T-1} e_t f_t.
\end{aligned}$$

From the analysis of assertion (i), we can see that

$$\begin{aligned}
e_t &= \sup_{l,k,p}|\hat{b}_{l,k,p} - b_{l,k,p}|O_p(1)\sum_{q=1}^{P}\sum_{(l,k)\in\mathcal{L}_p}\sqrt{2^{l-1}}\mathbf{I}(H_{l,2k-1}(t) \neq 1; H_{l,2k}(t) \neq 1) \\
&= O_p\Big(\sqrt{\frac{\log(T-1)}{n(T-1)}} + \lambda_{n,T}\Big)\sum_{q=1}^{P}\sum_{(l,k)\in\mathcal{L}_p}\sqrt{2^{l-1}}\mathbf{I}(H_{l,2k-1}(t) \neq 1; H_{l,2k}(t) \neq 1),
\end{aligned}$$

and

$$f_t = \sup_{(l,k)\in\mathcal{L}_p,p}|\hat{b}_{l,k,p}|O_p(1)\sum_{q=1}^{P}\sum_{(l,k)\in\mathcal{L}_p}\sqrt{2^{l-1}}\mathbf{I}(H_{l,2k-1}(t) \neq 1; H_{l,2k}(t) \neq 1).$$

Using Cauchy-Schwarz inequality to $(\sum_{q=1}^{P}\sum_{(l,k)\in\mathcal{L}_p}\sqrt{2^{l-1}}\mathbf{I}(H_{l,2k-1}(t)\neq 1; H_{l,2k}(t)\neq 1))^2$ over (l,k), we can infer that

$$e_t^2 \leq O_p\big(\frac{\log(T-1)}{n(T-1)}+\lambda_{n,T}^2\big)\sum_{q=1}^{P}\sum_{(l,k)\in\mathcal{L}_p}2^{l-1}\mathbf{I}(H_{l,2k-1}(t)\neq 1; H_{l,2k}(t)\neq 1),$$

and

$$\frac{1}{T-1}\sum_{t=2}^{T-2}f_t^2 \leq (\sup_{(l,k)\in\mathcal{L}_p,p}|\hat{b}_{l,k,p}|)^2 O_p(T-1).$$

If $\sqrt{T-1}\lambda_{n,T}\sim(\log(T-1)/n)^{\kappa/2}$, then $\text{plim}(\frac{1}{T-1}\sum_{t=2}^{T-2}f_t^2)=0$ as T and\or n pass to infinity, for any $\kappa\in]0,1[$.

Let us now examine the average of e_t^2 over t. If, in total, the maximal number of jumps is $S^*=\sum_p^P S_p$, then by Proposition 1 the number of non-zero coefficients is at most $(S^*+1)L$. By taking the average of e_t^2 over t, we can hence infer that

$$\frac{1}{T-1}\sum_{t=2}^{T-1}e_t^2 \leq O_p\big(\frac{\log(T-1)}{n(T-1)}+\lambda_{n,T}^2\big)\left(\min\{(S^*+1)\log(T-1),(T-1)\}\right).$$

Finally, because $\text{plim}(\frac{1}{T-1}\sum_{t=2}^{T-2}f_t^2)=0$, by Cauchy-Schwarz inequality, we can infer that $\frac{1}{T-1}\sum_{t=2}^{T-1}e_tf_t$ also can be neglected. Thus

$$\frac{1}{T-1}\sum_{t=2}^{T-1}(\tilde{\gamma}_{t,p}-\gamma_{t,p})^2 = O_p\big(\frac{J^*(\log(T-1)/n)^{\kappa}}{(T-1)}\big),$$

where $J^*=\min\{(S^*+1)\log(T-1),(T-1)\}$. This completes the proof. □

B.3 Proofs of Section 3

Proof of Lemma 2: We have to show that

$$\sup_{k,p\in\{1,\ldots,P\}}\left|\tilde{c}_{L,k,p}^{(m)}-c_{L,k,p}^{(m)}\right| = O_p\big(\sqrt{\log(T-1)/(n(T-1))}\big),$$

for $m=s,u$.

For $p\in\{1,\ldots,P\}$ and $m=s$, we have by construction

$$\begin{aligned}\tilde{c}_{L,k,p}^{(s)}-c_{L,k,p}^{(s)} &= \frac{1}{T-1}\sum_{t=2}^{T}\psi_{L,k}(t-1)(\tilde{\gamma}_{t,p}-\gamma_{t,p}),\\ &= \frac{1}{T-1}\sum_{t=2}^{T}\psi_{L,k}(t-1)\sum_{l,m,q}W_{l,m,p,q}(t)(\tilde{b}_{l,m,q}-b_{l,m,q}),\\ &= \frac{1}{T-1}\sum_{t\in\{\psi_{L,k}(t-1)\neq 0\}}\psi_{L,k}(t-1)\sum_{l,m,q}W_{l,m,p,q}(t)(\tilde{b}_{l,m,q}-b_{l,m,q}),\end{aligned}$$

where $\sum_{l,m,q}$ denotes the triple summation $\sum_{l=1}^{L}\sum_{k=1}^{K_l}\sum_{q=1}^{P}$.

Taking the absolute value, we obtain

$$|\tilde{c}^{(s)}_{L,k,p} - c^{(s)}_{L,k,p}| \leq \sup_{l,k,p} |\tilde{b}_{l,k,p} - b_{l,k,p}| \frac{1}{T-1} \sum_{t \in \{\psi_{L,k}(t-1) \neq 0\}} \left| \psi_{L,k}(t-1) \sum_{l,m,q} W_{l,m,p,q}(t) \right|.$$

Recall that $\frac{1}{T-1} \sum_{t \in \{\psi_{L,k}(t-1) \neq 0\}} \psi_{L,k}(t-1)^2 = 1$. By using Cauchy-Schwarz inequality, we can easily verify that

$$\frac{1}{T-1} \sum_{t \in \{\psi_{L,k}(t-1) \neq 0\}} \left| \psi_{L,k}(t-1) \sum_{l,m,q} W_{l,m,p,q}(t) \right| \leq \left(\frac{1}{T-1} \sum_{t \in \{\psi_{L,k}(t-1) \neq 0\}} (\sum_{l,m,q} W_{l,m,p,q}(t))^2 \right)^{1/2}.$$

Because the support of $\psi_{L,k}(t-1)$ is of length 2 ($\sum_t \mathbf{I}(t \in \{\psi_{L,k}(t-1) \neq 0\}) = 2$), by using a similar analysis to that used in the proof of Theorem 1, we can easily verify that the term in the last inequality is $O_p(1)$. By Lemma 1, we can hence infer that

$$|\tilde{c}^{(s)}_{L,k,p} - c^{(s)}_{L,k,p}| \leq \sup_{l,k,p} |\tilde{b}_{l,k,p} - b_{l,k,p}| O_p(1) = O_p(\sqrt{\log(T-1)/n(T-1)}).$$

The proof of $\sup_{L,k,p} |\tilde{c}^{(u)}_{L,k,p} - c^{(u)}_{L,k,p}|$ being $O_p(\sqrt{\log(T-1)/n(T-1)})$ is similar and thus omitted. □

Proof of Theorem 2: We begin with defining the following sets for each $p \in \{1, \ldots, \underline{P}\}$:

$$\begin{aligned}
\mathcal{J}_p &:= \{\tau_{1,p}, \ldots, \tau_{S_p,p}\},\\
\mathcal{J}^c_p &:= \{1, \ldots, T\} \setminus \mathcal{J}_p,\\
\overline{\mathcal{J}}_p &:= \{2, 4, \ldots, T-1\} \cap \mathcal{J}_p,\\
\underline{\mathcal{J}}_p &:= \{3, 5, \ldots, T\} \cap \mathcal{J}_p,\\
\overline{\mathcal{J}}^c_p &:= \{2, 4, \ldots, T-1\} \setminus \overline{\mathcal{J}}_p, \text{ and}\\
\underline{\mathcal{J}}^c_p &:= \{3, 5, \ldots, T\} \setminus \underline{\mathcal{J}}_p.
\end{aligned}$$

Here, $\mathcal{J}_p$ is the set of all jump locations for parameter $\beta_{t,p}$, $\mathcal{J}^c_p$ is its complement, which contains only the stability intervals, $\overline{\mathcal{J}}_p$ is the set of all even jump locations and $\underline{\mathcal{J}}_p$ is the set of all odd jump locations so that $\overline{\mathcal{J}}_p \cap \underline{\mathcal{J}}_p = \emptyset$ and $\overline{\mathcal{J}}_p \cup \underline{\mathcal{J}}_p = \mathcal{J}_p$. Finally, the sets $\overline{\mathcal{J}}^c_p$ and $\underline{\mathcal{J}}^c_p$ define the complements of $\overline{\mathcal{J}}_p$ and $\underline{\mathcal{J}}_p$, respectively.

Define the event

$$\omega_{n,T} := \{ \sup_{t \in \mathcal{J}^c_p,\, p \in \{1,\ldots,P\}} \{ |\Delta\tilde{\beta}^{(u)}_{t,p}| \mathbf{I}_{\overline{\mathcal{J}}^c_p} + |\Delta\tilde{\beta}^{(s)}_{t,p}| \mathbf{I}_{\underline{\mathcal{J}}^c_p} \} = 0 \},$$

where $\mathbf{I}_{\overline{\mathcal{J}}^c_p} = \mathbf{I}(t \in \overline{\mathcal{J}}^c_p)$, $\mathbf{I}_{\underline{\mathcal{J}}^c_p} = \mathbf{I}(t \in \underline{\mathcal{J}}^c_p)$ and $\mathbf{I}(.)$ is the indicator function.

To prove that no jump can be identified in the stability intervals, we have to show, that $P(\omega_{n,T}) \to 1$, if $\sqrt{\frac{n(T-1)}{\log(T-1)}} \lambda_{n,T} \to \infty$, as $n, T \to \infty$ or as $n \to \infty$ and T is fixed. Note that $\overline{\mathcal{J}}^c_p$ and $\underline{\mathcal{J}}^c_p$ are adjacent.

Let's now start with the no-jump case in $\overline{\mathcal{J}}_p^c$. By construction, we have, for all $t \in \{2, 4, \ldots, T-1\}$,

$$\Delta\tilde{\beta}_{t,p}^{(u)} = \sum_{k=1}^{K_L} \Delta\psi_{L,k}(t)\hat{c}_{L,k,p}^{(u)}$$

Recall that at $l = L$, the construction of the wavelets basis implies that at each $t \in \{2, 4, \ldots, T-1\}$ there is only one differenced basis $\Delta\psi_{L,k}(t)$ that is not zero. Let $\mathcal{K}_p^c = \{k|\Delta\psi_{L,k}(t) \neq 0,\, t \in \overline{\mathcal{J}}_p^c\} = \{k|\Delta\psi_{L,k}(t-1) \neq 0,\, t \in \underline{\mathcal{J}}_p^c\}$. We can infer that $\{\sup_{t\in\overline{\mathcal{J}}_p^c} |\sum_{k=1}^{K_L} \Delta\psi_{L,k}(t)\hat{c}_{L,k,p}^{(u)}| = 0\}$ occurs only if $\{\sup_{k\in\mathcal{K}_p^c} |c_{L,k,p}^{(u)}| = 0\}$ occurs.

By analogy, we can show the same assertion for the complement set $\underline{\mathcal{J}}_p^c$, i.e., $\{\sup_{t\in\underline{\mathcal{J}}_p^c} |\Delta\tilde{\beta}_{t,p}^{(s)}| = 0\}$ occurs only if $\{\sup_{k\in\mathcal{K}_p^c} |\hat{c}_{L,k,p}^{(s)}| = 0\}$ occurs.

To study $P(\omega_{n,T})$, it is hence sufficient to study

$$P(\sup_{k\in\mathcal{K}_p^c, m,p\in\{1,\ldots,P\}} |\hat{c}_{L,k,p}^{(m)}| = 0) = P(\sup_{k\in\mathcal{K}_p^c, m,p\in\{1,\ldots,P\}} |\tilde{c}_{L,k,p}^{(m)}| < \lambda_{n,T}).$$

By Lemma 2, $\sup_{k\in\mathcal{K}_p^c, m,p\in\{1,\ldots,P\}} |\tilde{c}_{L,k,p}^{(m)}| = O_p\big(\sqrt{\log(T-1)/n(T-1)}\big)$, since $c_{L,k,p}^{(m)} = 0$, for all $k \in \mathcal{K}_p^c$, and $p \in \{1, \ldots, P\}$. Thus, if $\sqrt{\frac{n(T-1)}{\log(T-1)}}\lambda_{n,T} \to \infty$, as $n, T \to \infty$ or $n \to \infty$ and T is fixed, then $P(\omega_{n,T}) \to 1$.

To complete the proof and demonstrate that all true jumps will be asymptotically identified, we suppose that there exists a jump location $\tau_{j,p} \in \overline{\mathcal{J}}_p \cup \underline{\mathcal{J}}_p$ for at least one $p \in \{1, \ldots, P\}$ that is not detected and show the contradiction. If $\tau_{j,p} \in \overline{\mathcal{J}}_p$, then

$$|\Delta\tilde{\beta}_{\tau_{j,p},p}^{(u)}|\mathbf{I}_{\overline{\mathcal{J}}_p} + |\Delta\tilde{\beta}_{\tau_{j,p},p}^{(s)}|\mathbf{I}_{\underline{\mathcal{J}}_p} = |\Delta\tilde{\beta}_{\tau_{j,p},p}^{(u)}|.$$

Adding and subtracting $\Delta\beta_{\tau_{j,p},p}^{(u)}$, we get

$$\begin{aligned}
\Delta\tilde{\beta}_{\tau_{j,p},p}^{(u)} &= \sum_{k=1}^{K_L} \Delta\psi_{L,k}(\tau_{j,p})(\tilde{c}_{L,k,p}^{(u)} - c_{L,k,p}^{(u)}) - \sum_{k=1}^{K_L} \Delta\psi_{L,k}(\tau_{j,p})\tilde{c}_{L,k,p}^{(u)}\mathbf{I}(\tilde{c}_{L,k,p}^{(u)} < \lambda_{n,T}) \\
&+ \sum_{k=1}^{K_L} \Delta\psi_{L,k}(\tau_{j,p})c_{L,k,p}^{(u)}, \\
&= I + II + III.
\end{aligned}$$

By Lemma 2, $I = o_p(1)$, $II = o_p(1)$ as long as $\sqrt{T-1}\lambda_{n,T} \to 0$, and $III \neq 0$ because $\sum_{k=1}^{K_L} \Delta\psi_{L,k}(t)c_{L,k,p}^{(u)} = \Delta\beta_{\tau_{j,p},p}^{(u)} \neq 0$. The probability of getting $\Delta\tilde{\beta}_{\tau_{j,p},p}^{(u)} = 0$ converges hence to zero.

If $\tau_{j,p} \in \underline{\mathcal{J}}_p$, then

$$|\Delta\tilde{\beta}_{\tau_{j,p},p}^{(u)}|\mathbf{I}_{\overline{\mathcal{J}}_p} + |\Delta\tilde{\beta}_{\tau_{j,p},p}^{(s)}|\mathbf{I}_{\underline{\mathcal{J}}_p} = |\Delta\tilde{\beta}_{\tau_{j,p},p}^{(s)}|.$$

The prove is similar to the case of $\tau_{j,p} \in \overline{\mathcal{J}}_p$ and thus omitted. This completes the proof. □

Proof of Theorem 3: Recall that the post-Wavelet estimator is obtained by replacing the set of the true jump locations $\tau_{1,1}, \ldots, \tau_{S_1+1,1}, \ldots, \tau_{1,P}, \ldots, \tau_{S_P+1,P}$ in $\hat{\beta}_{(\tau)} =$

$(\hat{\beta}_{\tau_{1,1}}, \ldots, \hat{\beta}_{\tau_{S_1+1,1}}, \ldots, \hat{\beta}_{\tau_{1,P}}, \ldots, \hat{\beta}_{\tau_{S_P+1,P}})'$ by the estimated jump locations $\tilde{\tau} := \{\tilde{\tau}_{j,p} | j \in \{1, \ldots, S_p+1\},\ p \in \{1, \ldots, P\}\}$, given $\tilde{S}_1 = S_1, \ldots, \tilde{S}_p = S_p$. By using Theorem 2, we can infer that, conditional on $\tilde{S}_1 = S_1, \ldots, \tilde{S}_p = S_p$,

$$\sqrt{n}\mathcal{T}^{\frac{1}{2}}_{(\tilde{\tau})}\hat{\beta}_{(\tilde{\tau})} = \sqrt{n}\mathcal{T}^{\frac{1}{2}}_{(\tau)}\hat{\beta}_{(\tau)} + o_p(1).$$

To study the asymptotic distribution of $\sqrt{n}\mathcal{T}^{\frac{1}{2}}_{(\tilde{\tau})}\hat{\beta}_{(\tilde{\tau})}$ it is hence sufficient to study $\sqrt{n}\mathcal{T}^{\frac{1}{2}}_{(\tau)}\hat{\beta}_{(\tau)}$.

$$\begin{aligned}\hat{\beta}_{(\tau)} &= \big(\sum_{i=1}^{n}\sum_{t=2}^{T} Z_{it,(\tau)}\tilde{\Delta}\dot{X}'_{it,(\tau)}\big)^{-1}\big(\sum_{i=1}^{n}\sum_{t=2}^{T} Z_{it,(\tau)}\Delta\dot{Y}_{it}\big)\\ &= \beta_{(\tau)} + \big(\sum_{i=1}^{n}\sum_{t=2}^{T} Z_{it,(\tau)}\tilde{\Delta}\dot{X}'_{it,(\tau)}\big)^{-1}\big(\sum_{i=1}^{n}\sum_{t=2}^{T} Z_{it,(\tau)}\Delta\dot{e}_{it}\big).\end{aligned}$$

Scaling by $\sqrt{n}\mathcal{T}^{\frac{1}{2}}_{(\tilde{\tau})}$ and rearranging, we get

$$\sqrt{n}\mathcal{T}^{\frac{1}{2}}_{(\tau)}\big(\hat{\beta}_{(\tilde{\tau})} - \beta_{(\tau)}\big) = \left((n\mathcal{T}_{(\tau)})^{-1}\sum_{i=1}^{n}\sum_{t=2}^{T} Z_{it,(\tau)}\tilde{\Delta}\dot{X}'_{it,(\tau)}\right)^{-1}\left((n\mathcal{T}_{(\tau)})^{-\frac{1}{2}}\sum_{i=1}^{n}\sum_{t=2}^{T} Z_{it,(\tau)}\Delta\dot{e}_{it}\right).$$

By Assumption E, the first term on the right hand side converges in probability to $Q^{\circ}_{(\tau)}$ and the second term converges in distribution to $N(0, V^{\circ}_{(\tau)})$. Slutsky's rule implies

$$\sqrt{n}\mathcal{T}^{\frac{1}{2}}_{(\tau)}\big(\hat{\beta}_{(\tau)} - \beta_{(\tau)}\big) \xrightarrow{d} N(0, (Q^{\circ}_{(\tau)})^{-1}\big(V^{\circ}_{(\tau)}\big)(Q^{\circ}_{(\tau)})^{-1}).$$

It follows

$$\sqrt{n}\mathcal{T}^{\frac{1}{2}}_{(\tilde{\tau})}\big(\hat{\beta}_{(\tilde{\tau})} - \beta_{(\tau)}\big) = \sqrt{n}\mathcal{T}^{\frac{1}{2}}_{(\tau)}\big(\hat{\beta}_{(\tau)} - \beta_{(\tau)}\big) + o_p(1) \xrightarrow{d} N(0, (Q^{\circ}_{(\tau)})^{-1}\big(V^{\circ}_{(\tau)}\big)(Q^{\circ}_{(\tau)})^{-1}).$$

This completes the Proof. □

Proof of Proposition 2 Consider $c = 1$ (the case of homoscedasticity without presence of auto- and cross-section correlation). Because by Assumption E, we know that

$$(n\mathcal{T}_{(\tau)})^{-1}\sum_{i=1}^{n}\sum_{t=2}^{T} Z_{it,(\tau)}\tilde{\Delta}\dot{X}'_{it,(\tau)} \xrightarrow{p} Q^{\circ}_{(\tau)} \quad \text{and}$$

$$(n\mathcal{T}_{(\tau)})^{-1}\sum_{i=1}^{n}\sum_{t=2}^{T}\sum_{j=1}^{n}\sum_{s=2}^{T} Z_{it,(\tau)}Z'_{js,(\tau)}\sigma_{ij,ts} \xrightarrow{p} V^{\circ}_{(\tau)},$$

it suffices to prove that

$$\hat{V}^{(1)}_{(\tilde{\tau})} = (n\mathcal{T}_{(\tilde{\tau})})^{-1}\sum_{i=1}^{n}\sum_{t=2}^{T} Z_{it,(\tilde{\tau})}Z_{it,(\tilde{\tau})}\hat{\sigma}^2 \xrightarrow{p} V^{(1)}_{(\tau)},$$

where $V^{(1)}_{(\tau)} = (n\mathcal{T}_{(\tau)})^{-1}\sum_{i=1}^n\sum_{t=2}^T Z_{it,(\tau)}Z'_{it,(\tau)}\sigma^2$, with $\sigma^2 = E_c(\Delta\dot{e}_{it})$.

$$
\begin{aligned}
\hat{V}^{(1)}_{(\tilde{\tau})} - V^{(1)}_{(\tau)} &= (\frac{1}{n(T-1)}\sum_{i=1}^n\sum_{t=2}^T \Delta\hat{e}^2_{it} - \sigma^2)(n\mathcal{T}_{(\tilde{\tau})})^{-1}\sum_{i=1}^n\sum_{t=2}^T Z_{it,(\tilde{\tau})}Z'_{it,(\tilde{\tau})},\\
&= +\sigma^2\left((n\mathcal{T}_{(\tilde{\tau})})^{-1}\sum_{i=1}^n\sum_{t=2}^T Z_{it,(\tilde{\tau})}Z'_{it,(\tilde{\tau})} - (n\mathcal{T}_{(\tau)})^{-1}\sum_{i=1}^n\sum_{t=2}^T Z_{it,(\tau)}Z'_{it,(\tau)}\right),\\
&= a + b.
\end{aligned}
$$

From Assumption B (ii), we can infer

$$
\begin{aligned}
||a|| &\leq (\frac{1}{n(T-1)}\sum_{i=1}^n\sum_{t=2}^T \Delta\hat{e}^2_{it} - \sigma^2)\frac{1}{n}\sum_{i=1}^n\sum_{t=2}^T ||(\mathcal{T}_{(\tilde{\tau})})^{-1/2}Z_{it,(\tilde{\tau})}||^2,\\
&= \left(\frac{1}{n(T-1)}\sum_{i=1}^n\sum_{t=2}^T((\Delta\hat{e}^2_{it} - \Delta e^2_{it}) + (\Delta e^2_{it} - \sigma^2))\right)\frac{1}{n}\sum_{i=1}^n\sum_{t=2}^T ||(\mathcal{T}_{(\tilde{\tau})})^{-1/2}Z_{it,(\tilde{\tau})}||^2.
\end{aligned}
$$

From

$$
\begin{aligned}
\Delta\hat{\dot{e}}_{it} &= \Delta\dot{Y}_{it} - \tilde{\Delta}\dot{X}'_{it,(\tilde{\tau})}\hat{\beta}_{(\tilde{\tau})},\\
&= \Delta\dot{e}_{it} + \tilde{\Delta}\dot{X}'_{it,(\tilde{\tau})}(\beta_{(\tilde{\tau})} - \hat{\beta}_{(\tilde{\tau})}),
\end{aligned}
\tag{22}
$$

and by using Theorem 3 together with Assumption B (ii), we can show that

$$
\frac{1}{n(T-1)}\sum_{i=1}^n\sum_{t=2}^T \Delta\hat{\dot{e}}_{it} - \frac{1}{n(T-1)}\sum_{i=1}^n\sum_{t=2}^T \Delta\dot{e}_{it} = o_p(1). \tag{23}
$$

By the law of large numbers,

$$
\frac{1}{n(T-1)}\sum_{i=1}^n\sum_{t=2}^T \Delta\dot{e}_{it} - \sigma^2 = o_p(1).
$$

Thus, $||a|| = (o_p(1) + o_p(1))O_p(1) = o_p(1)$. Moreover, from Theorem 2, we can infer that, given $\tilde{S}_1 = S_1, \ldots, \tilde{S}_P = S_P$,

$$
(n\mathcal{T}_{(\tilde{\tau})})^{-1}\sum_{i=1}^n\sum_{t=2}^T Z_{it,(\tilde{\tau})}Z_{it,(\tilde{\tau})} = (n\mathcal{T}_{(\tau)})^{-1}\sum_{i=1}^n\sum_{t=2}^T Z_{it,(\tau)}Z_{it,(\tau)} + o_p(1).
$$

Thus,

$$
\hat{V}^{(1)}_{(\tilde{\tau})} - V^{(1)}_{(\tau)} = o_p(1).
$$

Consider $c = 2$ (the case of cross-section heteroscedasticity without auto- and cross-section correlations). Because of Assumption E, it suffices to prove that

$$
\hat{V}^{(2)}_{(\tilde{\tau})} = (n\mathcal{T}_{(\tilde{\tau})})^{-1}\sum_{i=1}^n\sum_{t=2}^T Z_{it,(\tilde{\tau})}Z_{it,(\tilde{\tau})}\hat{\sigma}^2_i \xrightarrow{p} V^{(2)}_{(\tau)},
$$

where $V^{(2)}_{(\tau)} = (n\mathcal{T}_{(\tau)})^{-1}\sum_{i=1}^{n}\sum_{t=2}^{T} Z_{it,(\tau)}Z'_{it,(\tau)}\sigma_i^2$, with $\sigma_i^2 = E_c(\Delta\dot{e}_{it})$.

$$\begin{aligned}
\hat{V}^{(2)}_{(\tilde{\tau})} - V^{(2)}_{(\tau)} &= \frac{1}{n}\sum_{i=1}^{n}(\hat{\sigma}_i^2 - \sigma_i^2)(\mathcal{T}_{(\tilde{\tau})})^{-1}\sum_{t=2}^{T} Z_{it,(\tilde{\tau})}Z'_{it,(\tilde{\tau})},\\
&+ \frac{1}{n}\sum_{i=1}^{n}\sigma_i^2\left((\mathcal{T}_{(\tilde{\tau})})^{-1}\sum_{t=2}^{T} Z_{it,(\tilde{\tau})}Z'_{it,(\tilde{\tau})} - (\mathcal{T}_{(\tau)})^{-1}\sum_{t=2}^{T} Z_{it,(\tau)}Z'_{it,(\tau)}\right),\\
&= d + e.
\end{aligned}$$

$$\begin{aligned}
||d|| &\le \frac{1}{n}\sum_{i=1}^{n}(\hat{\sigma}_i^2 - \sigma_i^2)\sum_{t=2}^{T}||(\mathcal{T}_{(\tilde{\tau})})^{-1/2}Z_{it,(\tilde{\tau})}||^2,\\
&= \frac{1}{n}\sum_{i=1}^{n}((\hat{\sigma}_i^2 - \frac{1}{(T-1)}\sum_{t=2}^{T}\Delta\dot{e}_{it}) - (\sigma_i^2 - \frac{1}{(T-1)}\sum_{t=2}^{T}\Delta\dot{e}_{it})\frac{1}{n}\sum_{t=2}^{T}||(\mathcal{T}_{(\tilde{\tau})})^{-1/2}Z_{it,(\tilde{\tau})}||^2.
\end{aligned}$$

From Equation (22), and Theorem 3, we can infer

$$\frac{1}{(T-1)}\sum_{t=2}^{T}\Delta\hat{\dot{e}}_{it} - \frac{1}{(T-1)}\sum_{t=2}^{T}\Delta\dot{e}_{it} = o_p(1)\nu_i, \tag{24}$$

where $\frac{1}{n}\sum_{i=1}^{n}|\nu_i| = O_p(1)$. Moreover,

$$\sigma_i^2 - \frac{1}{(T-1)}\sum_{t=2}^{T}\Delta\dot{e}_{it} = o_p(1)\mu_i, \tag{25}$$

where $\frac{1}{n}\sum_{i=1}^{n}|\mu_i| = O_p(1)$. Note that the first terms in (24) and (25) do not depend on i. By using Assumption B (ii), we can infer

$$\begin{aligned}
||d|| &\le o_p(1)\frac{1}{n}\sum_{i=1}^{n}|\nu_i|\sum_{t=2}^{T}||(\mathcal{T}_{(\tilde{\tau})})^{-1/2}Z_{it,(\tilde{\tau})}||^2 + o_p(1)\frac{1}{n}\sum_{i=1}^{n}|\mu_i|\sum_{t=2}^{T}||(\mathcal{T}_{(\tilde{\tau})})^{-1/2}Z_{it,(\tilde{\tau})}||^2,\\
&= o_p(1)O_p(1) + o_p(1)O_p(1).
\end{aligned}$$

The proof of e being $o_p(1)$ is similar to the proof of b in the first part. This is because σ_i^2 does not affect the analysis.

The proof of $\hat{V}^{(3)}_{\tilde{\tau}}$ being $(n\mathcal{T}_{(\tau)})^{-1}\sum_{i=1}^{n}\sum_{t=2}^{T} Z_{it,(\tau)}Z'_{it,(\tau)}\sigma_t^2 + o_p(1)$, with $\sigma_t^2 = E_c(\Delta\dot{e}_{it})$ is conceptually similar and thus omitted.

Finally, consider $c = 4$ (The case of cross-section and time heteroscedasticity without auto- and cross-section correlations). As in the previous cases, all we need is to prove that

$$\hat{V}^{(4)}_{(\tilde{\tau})} = (n\mathcal{T}_{(\tilde{\tau})})^{-1}\sum_{i=1}^{n}\sum_{t=2}^{T} Z_{it,(\tilde{\tau})}Z'_{it,(\tilde{\tau})}\Delta\hat{\dot{e}}^2_{it} \xrightarrow{p} V^{(4)}_{(\tau)},$$

where

$$V^{(4)}_{(\tau)} = (n\mathcal{T}_{(\tau)})^{-1}\sum_{i=1}^{n}\sum_{t=2}^{T} Z_{it,(\tau)}Z'_{it,(\tau)}\sigma_{it}^2,$$

with $\sigma_{it}^2 = E_c(\Delta\dot{e}_{it})$.

$$\begin{aligned}\hat{V}_{(\tilde{\tau})}^{(4)} - V_{(\tau)}^{(4)} &= (n\mathcal{T}_{(\tau)})^{-1}\sum_{i=1}^{n}\sum_{t=2}^{T} Z_{it,(\tilde{\tau})}Z'_{it,(\tilde{\tau})}(\Delta\hat{\dot{e}}_{it}^2 - \Delta\dot{e}_{it}^2)\\ &+ (n\mathcal{T}_{(\tau)})^{-1}\sum_{i=1}^{n}\sum_{t=2}^{T}(Z_{it,(\tilde{\tau})}Z'_{it,(\tilde{\tau})} - Z_{it,(\tau)}Z'_{it,(\tau)})\Delta\dot{e}_{it}^2\\ &+ (n\mathcal{T}_{(\tau)})^{-1}\sum_{i=1}^{n}\sum_{t=2}^{T} Z_{it,(\tau)}Z'_{it,(\tau)}(\Delta\dot{e}_{it}^2 - \sigma_{it}^2).\\ &= f + g + h.\end{aligned}$$

Cauchy-Schwarz inequality implies

$$||f|| \leq \left((n\mathcal{T}_{(\tau)})^{-1}\sum_{i=1}^{n}\sum_{t=2}^{T}||Z_{it,(\tau)}||^2\right)^{1/2}\left((n\mathcal{T}_{(\tau)})^{-1}\sum_{i=1}^{n}\sum_{t=2}^{T}(\Delta\hat{\dot{e}}_{it}^2 - \Delta\dot{e}_{it}^2)\right)^{1/2} = o_p(1).$$

By using Theorem 3, we can also verify that $||g|| = o_p(1)$. Finally, Cauchy-Schwarz, Assumption B (ii), the law of large numbers implies that $||h|| = o_p(1)$. It follows

$$\hat{V}_{(\tilde{\tau})}^{(4)} \xrightarrow{p} V_{(\tau)}^{(4)}.$$

This completes the proof. □

C Simulation Results

			DGP2					
			Post-SAW			Qian and Su (2016)		
S	T	n	$\tilde{S}$	$\|\|\hat{\beta}-\beta\|\|^2/T$	HD/T	$\hat{S}$	$\|\|\hat{\beta}-\beta\|\|^2/T$	HD/T
1	33	30	1.320 (0.905)	0.625 (1.671)	0.079 (0.141)	1.038 (0.229)	0.000 (0.000)	0.009 (0.055)
1	33	60	1.052 (0.271)	0.013 (0.180)	0.010 (0.056)	1.000 (0.000)	0.000 (0.000)	0.000 (0.000)
1	33	120	1.010 (0.118)	0.000 (0.000)	0.001 (0.008)	1.000 (0.000)	0.000 (0.000)	0.000 (0.000)
1	33	300	1.002 (0.045)	0.000 (0.000)	0.000 (0.001)	1.000 (0.000)	0.000 (0.000)	0.000 (0.000)
1	65	30	1.494 (1.092)	0.766 (1.801)	0.114 (0.160)	1.002 (0.045)	0.000 (0.000)	0.000 (0.003)
1	65	60	1.122 (0.428)	0.012 (0.176)	0.020 (0.077)	1.000 (0.000)	0.000 (0.000)	0.000 (0.000)
1	65	120	1.006 (0.077)	0.000 (0.000)	0.000 (0.004)	1.000 (0.000)	0.000 (0.000)	0.000 (0.000)
1	65	300	1.000 (0.000)	0.000 (0.000)	0.000 (0.000)	1.000 (0.000)	0.000 (0.000)	0.000 (0.000)
1	129	30	2.432 (2.597)	0.613 (1.530)	0.202 (0.179)	1.000 (0.000)	0.000 (0.000)	0.000 (0.000)
1	129	60	1.196 (0.531)	0.000 (0.000)	0.032 (0.098)	1.000 (0.000)	0.000 (0.000)	0.000 (0.000)
1	129	120	1.026 (0.213)	0.000 (0.000)	0.003 (0.032)	1.000 (0.000)	0.000 (0.000)	0.000 (0.000)
1	129	300	1.000 (0.000)	0.000 (0.000)	0.000 (0.000)	1.000 (0.000)	0.000 (0.000)	0.000 (0.000)
2	33	30	2.216 (1.771)	0.930 (1.565)	0.106 (0.140)	2.046 (0.237)	0.000 (0.000)	0.005 (0.029)
2	33	60	2.060 (0.304)	0.019 (0.183)	0.010 (0.049)	2.002 (0.045)	0.000 (0.000)	0.000 (0.005)
2	33	120	2.014 (0.134)	0.000 (0.000)	0.001 (0.009)	2.000 (0.000)	0.000 (0.000)	0.000 (0.000)
2	33	300	2.000 (0.000)	0.000 (0.000)	0.000 (0.000)	2.000 (0.000)	0.000 (0.000)	0.000 (0.000)
2	65	30	2.448 (2.458)	1.158 (1.679)	0.137 (0.141)	2.006 (0.077)	0.000 (0.000)	0.001 (0.014)
2	65	60	2.072 (0.389)	0.033 (0.244)	0.013 (0.055)	2.000 (0.000)	0.000 (0.000)	0.000 (0.000)
2	65	120	2.006 (0.077)	0.000 (0.000)	0.000 (0.004)	2.000 (0.000)	0.000 (0.000)	0.000 (0.000)
2	65	300	2.000 (0.000)	0.000 (0.000)	0.000 (0.000)	2.000 (0.000)	0.000 (0.000)	0.000 (0.000)
2	129	30	3.100 (3.729)	1.209 (1.645)	0.166 (0.129)	2.000 (0.000)	0.000 (0.000)	0.000 (0.000)
2	129	60	2.196 (0.609)	0.027 (0.215)	0.020 (0.064)	2.000 (0.000)	0.000 (0.000)	0.000 (0.000)
2	129	120	2.020 (0.166)	0.000 (0.000)	0.001 (0.014)	2.000 (0.000)	0.000 (0.000)	0.000 (0.000)
2	129	300	2.000 (0.000)	0.000 (0.000)	0.000 (0.000)	2.000 (0.000)	0.000 (0.000)	0.000 (0.000)
3	33	30	3.152 (1.599)	0.967 (1.385)	0.107 (0.123)	3.050 (0.252)	0.000 (0.000)	0.004 (0.021)
3	33	60	3.068 (0.316)	0.015 (0.137)	0.007 (0.033)	3.004 (0.063)	0.000 (0.000)	0.000 (0.003)
3	33	120	3.006 (0.077)	0.000 (0.000)	0.000 (0.004)	3.000 (0.000)	0.000 (0.000)	0.000 (0.000)
3	33	300	3.000 (0.000)	0.000 (0.000)	0.000 (0.000)	3.000 (0.000)	0.000 (0.000)	0.000 (0.000)
3	65	30	3.482 (2.920)	1.156 (1.470)	0.129 (0.124)	3.010 (0.100)	0.000 (0.000)	0.002 (0.016)
3	65	60	3.106 (0.418)	0.014 (0.139)	0.009 (0.036)	3.000 (0.000)	0.000 (0.000)	0.000 (0.000)
3	65	120	3.014 (0.148)	0.000 (0.000)	0.000 (0.003)	3.000 (0.000)	0.000 (0.000)	0.000 (0.000)
3	65	300	3.000 (0.000)	0.000 (0.000)	0.000 (0.000)	3.000 (0.000)	0.000 (0.000)	0.000 (0.000)
3	129	30	4.296 (5.102)	1.228 (1.483)	0.145 (0.105)	3.000 (0.000)	0.000 (0.000)	0.000 (0.000)
3	129	60	3.174 (0.879)	0.042 (0.236)	0.019 (0.056)	3.000 (0.000)	0.000 (0.000)	0.000 (0.000)
3	129	120	3.028 (0.208)	0.000 (0.000)	0.002 (0.016)	3.000 (0.000)	0.000 (0.000)	0.000 (0.000)
3	129	300	3.000 (0.000)	0.000 (0.000)	0.000 (0.000)	3.000 (0.000)	0.000 (0.000)	0.000 (0.000)

			DGP3					
			Post-SAW			Qian and Su (2016)		
S	T	n	$\tilde{S}$	$\|\|\hat{\beta}-\beta\|\|^2/T$	HD/T	$\hat{S}$	$\|\|\hat{\beta}-\beta\|\|^2/T$	HD/T
1	33	30	0.996 (0.063)	0.028 (0.346)	0.000 (0.000)	2.254 (1.781)	0.012 (0.012)	0.150 (0.177)
1	33	60	0.992 (0.089)	0.025 (0.248)	0.000 (0.000)	1.290 (0.671)	0.003 (0.004)	0.051 (0.119)
1	33	120	1.000 (0.000)	0.002 (0.002)	0.000 (0.000)	1.046 (0.237)	0.001 (0.001)	0.008 (0.045)
1	33	300	1.000 (0.000)	0.001 (0.001)	0.000 (0.000)	1.006 (0.077)	0.000 (0.000)	0.001 (0.020)
1	65	30	0.998 (0.045)	0.014 (0.244)	0.000 (0.000)	1.390 (0.982)	0.004 (0.005)	0.059 (0.131)
1	65	60	0.988 (0.109)	0.035 (0.303)	0.000 (0.000)	1.036 (0.197)	0.001 (0.001)	0.009 (0.055)
1	65	120	1.000 (0.000)	0.001 (0.001)	0.000 (0.000)	1.004 (0.063)	0.000 (0.000)	0.001 (0.015)
1	65	300	1.000 (0.000)	0.000 (0.000)	0.000 (0.000)	1.002 (0.045)	0.000 (0.000)	0.000 (0.006)
1	129	30	0.992 (0.089)	0.045 (0.486)	0.000 (0.000)	1.042 (0.237)	0.001 (0.001)	0.010 (0.054)
1	129	60	0.990 (0.100)	0.029 (0.277)	0.000 (0.000)	1.006 (0.077)	0.000 (0.000)	0.001 (0.013)
1	129	120	0.998 (0.045)	0.004 (0.080)	0.000 (0.000)	1.002 (0.045)	0.000 (0.000)	0.000 (0.000)
1	129	300	1.000 (0.000)	0.000 (0.000)	0.000 (0.000)	1.000 (0.000)	0.000 (0.000)	0.000 (0.000)
2	33	30	2.000 (0.000)	0.010 (0.008)	0.000 (0.000)	3.196 (1.647)	0.013 (0.011)	0.089 (0.108)
2	33	60	1.986 (0.118)	0.032 (0.222)	0.005 (0.039)	2.238 (0.631)	0.003 (0.004)	0.021 (0.059)
2	33	120	1.998 (0.045)	0.004 (0.050)	0.001 (0.015)	2.054 (0.267)	0.001 (0.001)	0.006 (0.032)
2	33	300	2.900 (0.000)	0.001 (0.001)	0.000 (0.000)	2.002 (0.045)	0.000 (0.000)	0.000 (0.007)
2	65	30	1.994 (0.077)	0.026 (0.272)	0.002 (0.025)	2.350 (0.872)	0.004 (0.005)	0.030 (0.076)
2	65	60	1.976 (0.153)	0.047 (0.282)	0.008 (0.049)	2.074 (0.317)	0.001 (0.001)	0.007 (0.036)
2	65	120	1.998 (0.045)	0.004 (0.054)	0.001 (0.014)	2.010 (0.100)	0.001 (0.001)	0.001 (0.017)
2	65	300	2.900 (0.000)	0.000 (0.000)	0.000 (0.000)	2.000 (0.000)	0.000 (0.000)	0.000 (0.000)
2	129	30	1.988 (0.109)	0.046 (0.397)	0.004 (0.036)	2.064 (0.310)	0.001 (0.002)	0.008 (0.040)
2	129	60	1.958 (0.201)	0.079 (0.372)	0.014 (0.067)	2.000 (0.000)	0.001 (0.000)	0.000 (0.000)
2	129	120	1.992 (0.089)	0.010 (0.106)	0.003 (0.030)	2.000 (0.000)	0.000 (0.000)	0.000 (0.000)
2	129	300	2.000 (0.000)	0.000 (0.000)	0.000 (0.000)	2.000 (0.000)	0.000 (0.000)	0.000 (0.000)
3	33	30	2.998 (0.045)	0.018 (0.123)	0.000 (0.011)	4.272 (1.560)	0.015 (0.012)	0.067 (0.078)
3	33	60	2.994 (0.077)	0.014 (0.104)	0.001 (0.019)	3.378 (0.707)	0.005 (0.004)	0.027 (0.053)
3	33	120	3.000 (0.000)	0.003 (0.002)	0.000 (0.000)	3.058 (0.258)	0.002 (0.001)	0.005 (0.025)
3	33	300	3.000 (0.000)	0.001 (0.001)	0.000 (0.000)	3.004 (0.063)	0.001 (0.000)	0.000 (0.005)
3	65	30	2.988 (0.126)	0.035 (0.291)	0.003 (0.031)	3.484 (1.026)	0.006 (0.005)	0.030 (0.061)
3	65	60	2.974 (0.159)	0.039 (0.221)	0.006 (0.039)	3.068 (0.289)	0.002 (0.001)	0.005 (0.026)
3	65	120	2.998 (0.045)	0.003 (0.040)	0.000 (0.011)	3.008 (0.089)	0.001 (0.001)	0.001 (0.007)
3	65	300	3.000 (0.000)	0.001 (0.000)	0.000 (0.000)	3.000 (0.000)	0.000 (0.000)	0.000 (0.000)
3	129	30	2.994 (0.077)	0.019 (0.209)	0.001 (0.019)	3.082 (0.340)	0.002 (0.002)	0.007 (0.032)
3	129	60	2.948 (0.231)	0.072 (0.307)	0.013 (0.057)	3.012 (0.126)	0.001 (0.001)	0.001 (0.008)
3	129	120	2.992 (0.089)	0.008 (0.079)	0.002 (0.022)	3.002 (0.045)	0.000 (0.000)	0.000 (0.001)
3	129	300	3.000 (0.000)	0.000 (0.000)	0.000 (0.000)	3.000 (0.000)	0.000 (0.000)	0.000 (0.000)

			DGP4					
			Post-SAW			Qian and Su (2016)		
S	T	n	$\tilde{S}$	$\|\|\hat{\beta}-\beta\|\|^2/T$	HD$/T$	$\hat{S}$	$\|\|\hat{\beta}-\beta\|\|^2/T$	HD$/T$
1	33	30	0.996 (0.063)	0.027 (0.347)	0.000 (0.000)	5.764 (3.071)	0.045 (0.026)	0.347 (0.151)
1	33	60	0.996 (0.063)	0.014 (0.176)	0.000 (0.000)	2.818 (2.066)	0.013 (0.011)	0.201 (0.187)
1	33	120	1.000 (0.000)	0.001 (0.001)	0.000 (0.000)	1.418 (0.844)	0.003 (0.003)	0.072 (0.139)
1	33	300	1.000 (0.000)	0.001 (0.001)	0.000 (0.000)	1.026 (0.171)	0.001 (0.001)	0.004 (0.035)
1	65	30	0.996 (0.063)	0.025 (0.345)	0.000 (0.000)	4.132 (3.586)	0.020 (0.018)	0.235 (0.197)
1	65	60	0.984 (0.126)	0.046 (0.349)	0.000 (0.000)	1.562 (1.173)	0.003 (0.005)	0.081 (0.154)
1	65	120	1.000 (0.000)	0.001 (0.001)	0.000 (0.000)	1.100 (0.350)	0.001 (0.001)	0.020 (0.081)
1	65	300	1.000 (0.000)	0.000 (0.000)	0.000 (0.000)	1.002 (0.045)	0.000 (0.000)	0.000 (0.006)
1	129	30	0.992 (0.089)	0.045 (0.486)	0.000 (0.000)	1.640 (1.491)	0.004 (0.006)	0.081 (0.151)
1	129	60	0.980 (0.140)	0.056 (0.389)	0.000 (0.000)	1.060 (0.262)	0.001 (0.001)	0.014 (0.070)
1	129	120	0.998 (0.045)	0.004 (0.080)	0.000 (0.000)	1.016 (0.141)	0.000 (0.000)	0.005 (0.046)
1	129	300	1.000 (0.000)	0.000 (0.000)	0.000 (0.000)	1.000 (0.000)	0.000 (0.000)	0.000 (0.000)
2	33	30	1.994 (0.077)	0.029 (0.277)	0.002 (0.026)	6.778 (3.004)	0.047 (0.025)	0.216 (0.096)
2	33	60	1.994 (0.077)	0.015 (0.145)	0.002 (0.026)	3.728 (1.881)	0.014 (0.011)	0.138 (0.120)
2	33	120	2.000 (0.000)	0.002 (0.002)	0.000 (0.000)	2.356 (0.755)	0.003 (0.004)	0.038 (0.082)
2	33	300	2.000 (0.000)	0.001 (0.001)	0.000 (0.000)	2.046 (0.228)	0.001 (0.001)	0.006 (0.031)
2	65	30	1.998 (0.045)	0.011 (0.166)	0.001 (0.014)	5.128 (3.252)	0.021 (0.016)	0.160 (0.125)
2	65	60	1.982 (0.133)	0.035 (0.243)	0.006 (0.043)	2.592 (1.173)	0.004 (0.005)	0.052 (0.097)
2	65	120	1.998 (0.045)	0.003 (0.052)	0.001 (0.014)	2.086 (0.350)	0.001 (0.001)	0.010 (0.044)
2	65	300	2.000 (0.000)	0.000 (0.000)	0.000 (0.000)	2.000 (0.000)	0.000 (0.000)	0.000 (0.000)
2	129	30	1.994 (0.077)	0.024 (0.282)	0.002 (0.026)	2.730 (1.694)	0.005 (0.007)	0.048 (0.092)
2	129	60	1.956 (0.215)	0.080 (0.379)	0.013 (0.065)	2.116 (0.413)	0.001 (0.001)	0.014 (0.051)
2	129	120	1.996 (0.063)	0.005 (0.075)	0.001 (0.021)	2.012 (0.126)	0.001 (0.000)	0.001 (0.014)
2	129	300	2.000 (0.000)	0.000 (0.000)	0.000 (0.000)	2.000 (0.000)	0.000 (0.000)	0.000 (0.000)
3	33	30	2.992 (0.089)	0.033 (0.240)	0.002 (0.022)	7.476 (2.854)	0.048 (0.025)	0.153 (0.071)
3	33	60	2.990 (0.100)	0.019 (0.139)	0.002 (0.024)	4.590 (1.716)	0.014 (0.010)	0.084 (0.080)
3	33	120	2.998 (0.045)	0.004 (0.039)	0.000 (0.011)	3.444 (0.858)	0.004 (0.004)	0.031 (0.060)
3	33	300	3.000 (0.000)	0.001 (0.001)	0.000 (0.000)	3.068 (0.296)	0.001 (0.001)	0.007 (0.030)
3	65	30	2.998 (0.045)	0.011 (0.120)	0.000 (0.011)	6.064 (3.310)	0.022 (0.017)	0.107 (0.086)
3	65	60	2.966 (0.181)	0.050 (0.251)	0.008 (0.045)	3.610 (1.160)	0.005 (0.005)	0.037 (0.065)
3	65	120	2.998 (0.045)	0.003 (0.040)	0.000 (0.011)	3.104 (0.360)	0.002 (0.002)	0.009 (0.033)
3	65	300	3.000 (0.000)	0.001 (0.000)	0.000 (0.000)	3.006 (0.077)	0.001 (0.000)	0.001 (0.009)
3	129	30	2.982 (0.133)	0.052 (0.364)	0.004 (0.033)	3.766 (1.589)	0.006 (0.006)	0.040 (0.071)
3	129	60	2.968 (0.176)	0.046 (0.244)	0.008 (0.044)	3.138 (0.451)	0.002 (0.002)	0.010 (0.037)
3	129	120	2.986 (0.118)	0.013 (0.105)	0.003 (0.029)	3.016 (0.126)	0.001 (0.001)	0.002 (0.015)
3	129	300	3.000 (0.000)	0.000 (0.000)	0.000 (0.000)	3.000 (0.000)	0.000 (0.000)	0.000 (0.000)

DGP5

			Post-SAW			Qian and Su (2016)		
S	T	n	$\tilde{S}$	$\|\|\hat{\beta}-\beta\|\|^2/T$	HD/T	$\hat{S}$	$\|\|\hat{\beta}-\beta\|\|^2/T$	HD/T
1	33	30	1.000 (0.000)	0.005 (0.004)	0.000 (0.000)	4.296 (2.382)	0.037 (0.026)	0.240 (0.144)
1	33	60	1.000 (0.000)	0.003 (0.003)	0.000 (0.000)	2.558 (1.778)	0.012 (0.012)	0.131 (0.138)
1	33	120	1.000 (0.000)	0.001 (0.001)	0.000 (0.000)	1.736 (1.266)	0.004 (0.006)	0.064 (0.110)
1	33	300	1.000 (0.000)	0.000 (0.000)	0.000 (0.000)	1.188 (0.527)	0.001 (0.001)	0.014 (0.052)
1	65	30	1.000 (0.000)	0.002 (0.002)	0.000 (0.000)	4.242 (3.333)	0.021 (0.020)	0.206 (0.168)
1	65	60	1.000 (0.000)	0.001 (0.001)	0.000 (0.000)	2.092 (1.985)	0.005 (0.008)	0.091 (0.143)
1	65	120	1.000 (0.000)	0.001 (0.001)	0.000 (0.000)	1.330 (0.940)	0.001 (0.003)	0.033 (0.093)
1	65	300	1.000 (0.000)	0.000 (0.000)	0.000 (0.000)	1.014 (0.184)	0.000 (0.000)	0.001 (0.017)
1	129	30	1.000 (0.000)	0.001 (0.001)	0.000 (0.000)	3.008 (3.409)	0.009 (0.013)	0.126 (0.176)
1	129	60	1.000 (0.000)	0.001 (0.001)	0.000 (0.000)	1.474 (1.694)	0.002 (0.004)	0.034 (0.100)
1	129	120	1.000 (0.000)	0.000 (0.000)	0.000 (0.000)	1.074 (0.391)	0.000 (0.001)	0.009 (0.054)
1	129	300	1.000 (0.000)	0.000 (0.000)	0.000 (0.000)	1.000 (0.000)	0.000 (0.000)	0.000 (0.000)
2	33	30	2.000 (0.000)	0.007 (0.006)	0.000 (0.000)	5.250 (2.389)	0.037 (0.026)	0.131 (0.079)
2	33	60	2.000 (0.000)	0.004 (0.003)	0.000 (0.000)	3.746 (1.887)	0.013 (0.012)	0.080 (0.077)
2	33	120	2.000 (0.000)	0.002 (0.001)	0.000 (0.000)	2.612 (1.142)	0.004 (0.005)	0.033 (0.056)
2	33	300	2.000 (0.000)	0.001 (0.001)	0.000 (0.000)	2.112 (0.490)	0.001 (0.001)	0.007 (0.030)
2	65	30	2.000 (0.000)	0.004 (0.003)	0.000 (0.000)	5.310 (3.286)	0.024 (0.020)	0.115 (0.095)
2	65	60	2.000 (0.000)	0.002 (0.002)	0.000 (0.000)	3.236 (1.957)	0.007 (0.008)	0.054 (0.079)
2	65	120	2.000 (0.000)	0.001 (0.001)	0.000 (0.000)	2.296 (0.931)	0.002 (0.003)	0.015 (0.044)
2	65	300	2.000 (0.000)	0.000 (0.000)	0.000 (0.000)	2.034 (0.255)	0.000 (0.001)	0.002 (0.016)
2	129	30	2.000 (0.000)	0.002 (0.001)	0.000 (0.000)	4.046 (3.487)	0.010 (0.014)	0.074 (0.103)
2	129	60	2.000 (0.000)	0.001 (0.001)	0.000 (0.000)	2.424 (1.311)	0.002 (0.004)	0.022 (0.060)
2	129	120	2.000 (0.000)	0.000 (0.000)	0.000 (0.000)	2.058 (0.327)	0.001 (0.001)	0.004 (0.023)
2	129	300	2.000 (0.000)	0.000 (0.000)	0.000 (0.000)	2.000 (0.000)	0.000 (0.000)	0.000 (0.000)
3	33	30	3.000 (0.000)	0.009 (0.006)	0.000 (0.000)	5.510 (1.696)	0.037 (0.024)	0.074 (0.062)
3	33	60	3.000 (0.000)	0.005 (0.003)	0.000 (0.000)	4.400 (1.237)	0.015 (0.011)	0.036 (0.037)
3	33	120	3.000 (0.000)	0.002 (0.001)	0.000 (0.000)	3.770 (0.855)	0.006 (0.005)	0.021 (0.026)
3	33	300	3.000 (0.000)	0.001 (0.001)	0.000 (0.000)	3.262 (0.523)	0.002 (0.002)	0.007 (0.014)
3	65	30	3.000 (0.000)	0.005 (0.003)	0.000 (0.000)	6.176 (3.184)	0.024 (0.020)	0.072 (0.057)
3	65	60	3.000 (0.000)	0.002 (0.002)	0.000 (0.000)	4.110 (1.902)	0.007 (0.008)	0.031 (0.044)
3	65	120	3.000 (0.000)	0.001 (0.001)	0.000 (0.000)	3.356 (0.957)	0.002 (0.003)	0.012 (0.029)
3	65	300	3.000 (0.000)	0.000 (0.000)	0.000 (0.000)	3.026 (0.264)	0.000 (0.001)	0.001 (0.010)
3	129	30	3.000 (0.000)	0.002 (0.002)	0.000 (0.000)	5.150 (3.469)	0.011 (0.013)	0.055 (0.072)
3	129	60	3.000 (0.000)	0.001 (0.001)	0.000 (0.000)	3.408 (1.154)	0.002 (0.003)	0.016 (0.042)
3	129	120	3.000 (0.000)	0.001 (0.000)	0.000 (0.000)	3.056 (0.370)	0.001 (0.001)	0.003 (0.018)
3	129	300	3.000 (0.000)	0.000 (0.000)	0.000 (0.000)	3.004 (0.089)	0.000 (0.000)	0.000 (0.005)

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